Transversal AND in Quantum Codes

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A quantum AND gate

Problem: the **AND** gate is not reversible on bits and qubits

AND: $(a, b) \mapsto ab$ $(0, 0), (0, 1), (1, 0) \mapsto 0$ $(1, 1) \mapsto 1$

For qubit computation:

Need at least 3 qubits to compute AND — the Toffoli gate: $(a, b, c) \mapsto (a, b, c \oplus ab)$

However… what if there's a third level? **Idea**: each qu*bit* is natively a qu*trit*.

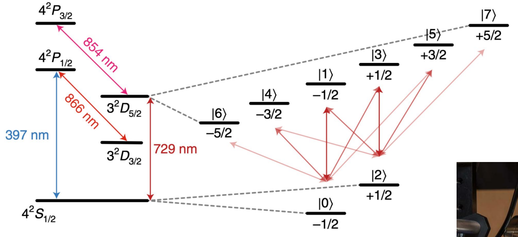
This talk: (1) a quantum AND gate with just 2 qutrits

(2) quantum codes with a fault-tolerant (transversal) AND

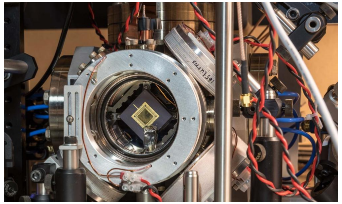
What are qutrits and why do we care about them?

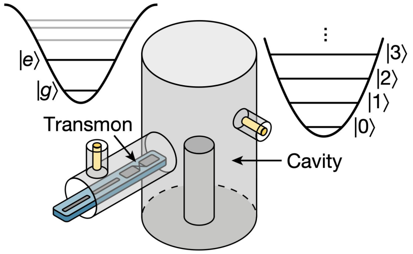
Qutrits: ternary (3-level) quantum system

1. More efficient algorithms & compilation than qubits (2-level)
2. Quantum computing hardwares are natively multi-level
3. **Better quantum error-correcting (QEC) codes** than what is possible for qubits

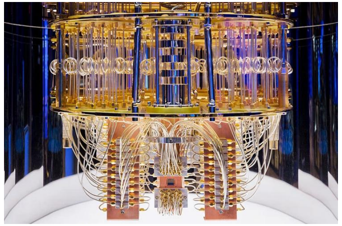


Schematic of trapped ion device [Rin+22]





Schematic of superconducting device [Bro+25]



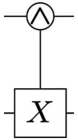
How to compute with qutrits?

Basis states: $|0\rangle, |1\rangle, |2\rangle$

Pauli X ("shift"): $X\,|k\rangle = |k+1 \pmod 3\rangle$

Pauli Z ("clock"): $Z\,|k\rangle = \omega^k\,|k\rangle$

CX:



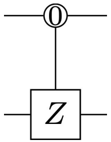
$|i, j\rangle \mapsto |i, i+j \pmod 3\rangle$

<u>Clifford group = <H, S, CX></u>

T: $T := Z(\tfrac{1}{3}, -\tfrac{1}{3}) = \mathrm{diag}(1, e^{\frac{2\pi i}{9}}, e^{\frac{-2\pi i}{9}})$

$|0\rangle$-controlled Z:

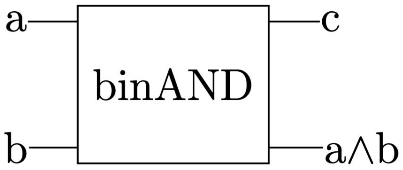
$$|0\rangle \otimes |\psi\rangle \mapsto |0\rangle \otimes Z|\psi\rangle,$$
$$|1\rangle \otimes |\psi\rangle \mapsto |1\rangle \otimes |\psi\rangle$$
$$|2\rangle \otimes |\psi\rangle \mapsto |2\rangle \otimes |\psi\rangle$$



(T-count 3)

<u>Non-Clifford</u>

Qutrit emulation of binary AND

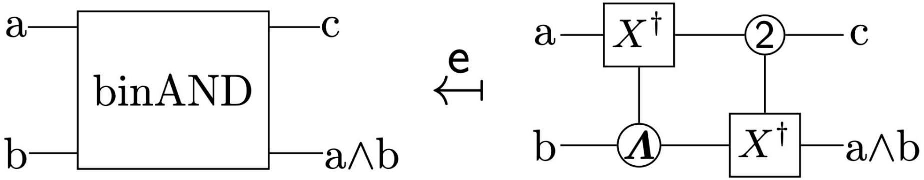


a	b		a∧b
0	0		0
0	1		0
1	0		0
1	1		1

Definition: a qutrit emulation of a qubit gate is when the behavior of the first two levels match but the third level is left unconstrained.

Qutrit circuit emulating binary AND gate with its truth table.

Qutrit emulation of binary AND

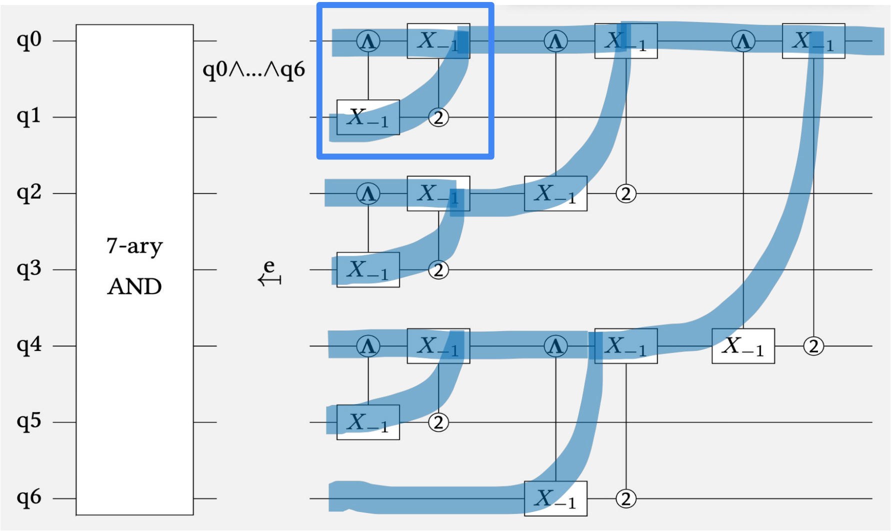


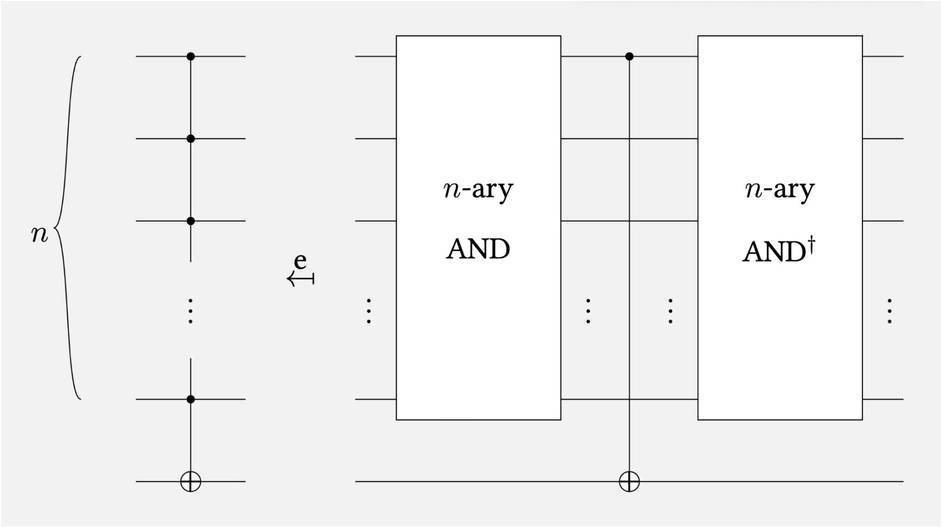
Definition: a qutrit emulation of a qubit gate is when the behavior of the first two levels match but the third level is left unconstrained.

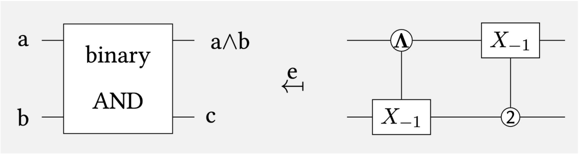
a	b	c	a∧b
0	0	0	0
0	1	2	0
0	2	1	2
1	0	1	0
1	1	0	1
1	2	2	1
2	0	2	2
2	1	1	1
2	2	0	2

Qutrit circuit emulating binary AND gate with its truth table.

Emulate AND with qutrits = width and depth savings



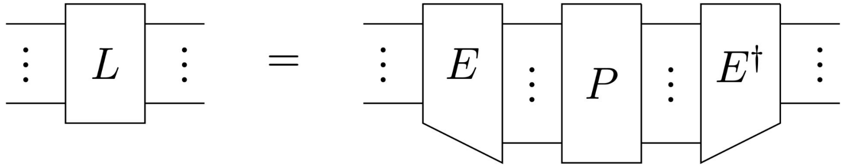




Roadmap

1. Computing AND using two qutrits
2. **Quantum codes with a transversal AND gate**
   a. **A symmetric T depth-one circuit for AND $\rightarrow$ $[[3, 2, 1]]_3$ code**
   b. Improving $[[3, 2, 1]]_3 \rightarrow [[6, 2, 2]]_3$ code
   c. Concatenating $[[6, 2, 2]]_3 \rightarrow [[48, 2, 4]]_3$ code
3. Qubit subspace projection and magic state injection

The **$[[n, k, d]]_3$** quantum error-correcting code:
Encodes n physical qutrits into k logical qutrits with code distance d.



Logical gate

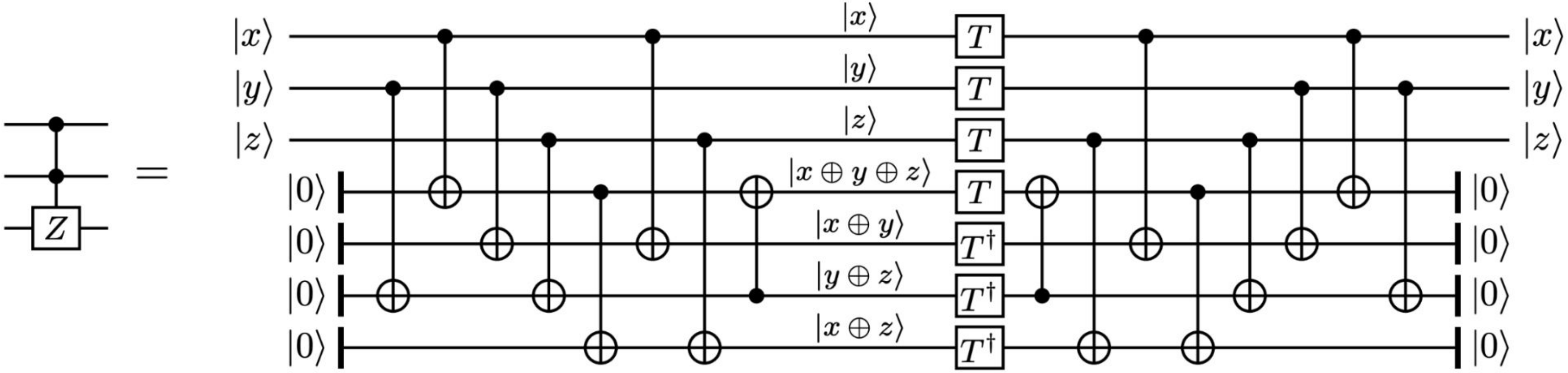
Physical implementation of the gate

Code distance d = minimum weight error that can go undetected by the code

d=1 codes are useless: detect and correct nothing
d=2 codes detect, but don't correct, errors
d=3 codes detect and correct single-qubit errors

distance d codes detect up to d-1 errors and correct up to (d-1)/2 errors

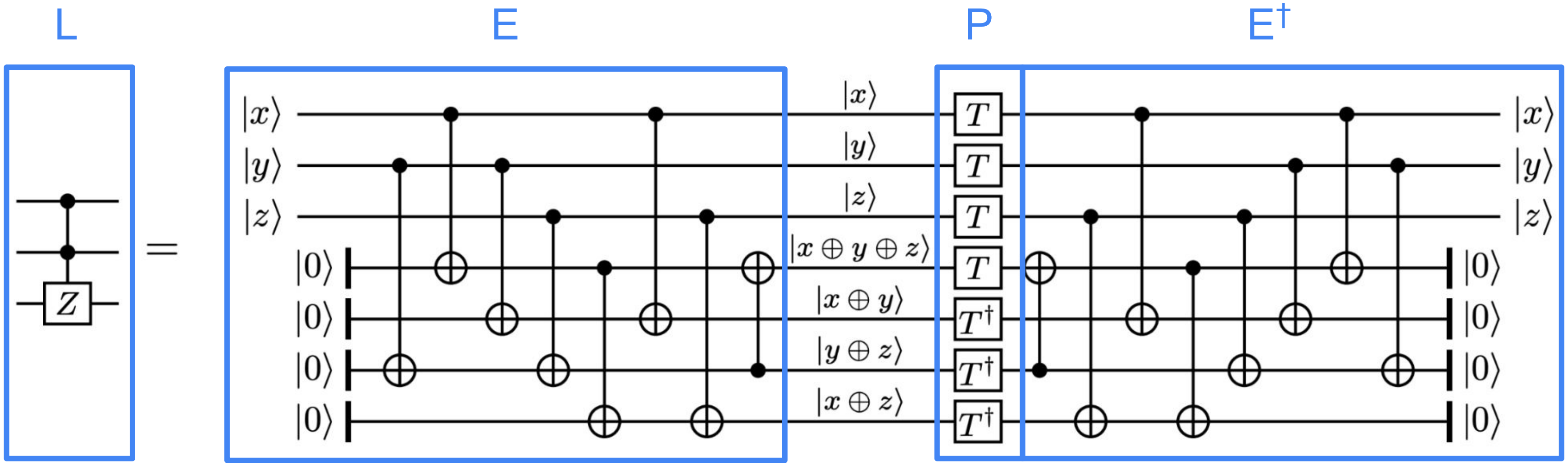
Consider the following symmetric T-depth 1 circuit:

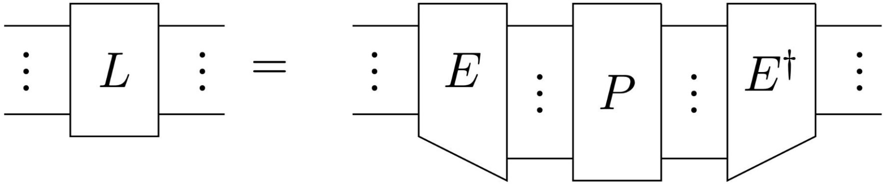


Circuit from [Selinger13]

This is a Clifford+T circuit for the CCZ gate.

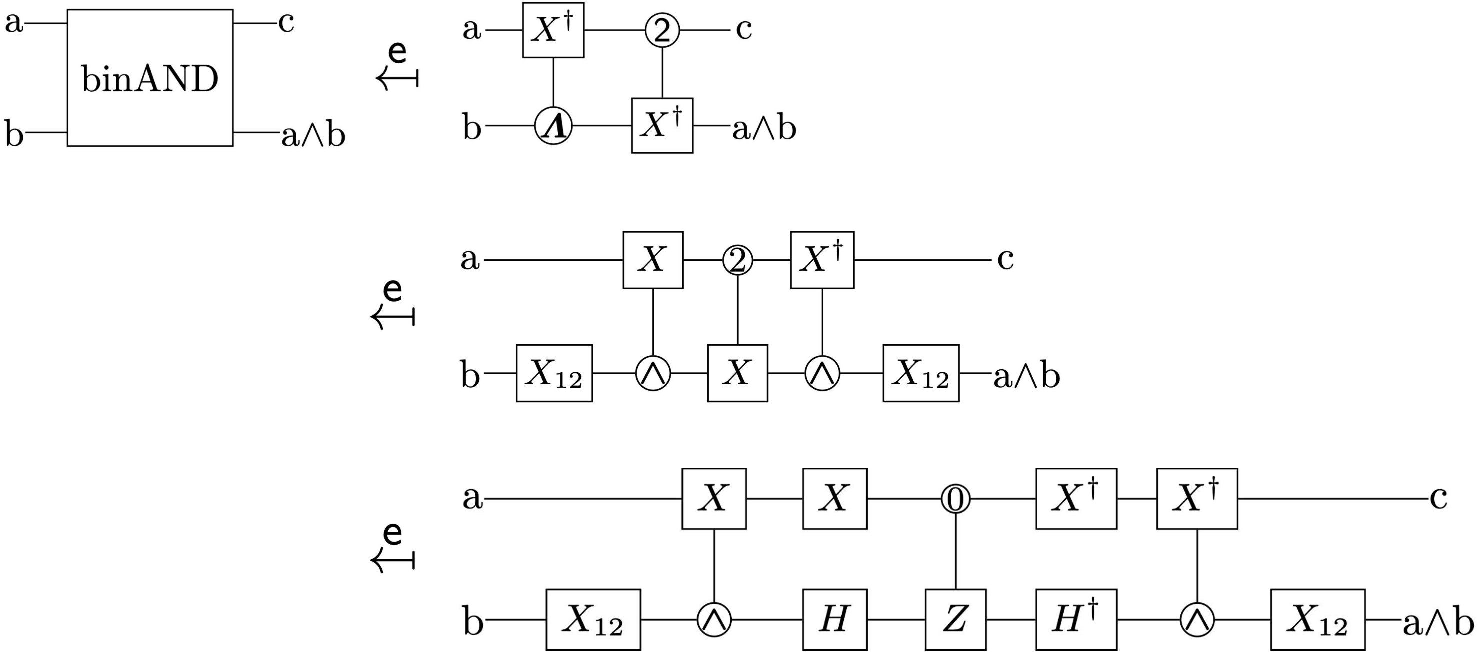
A symmetric T-depth 1 circuit defines a QEC code.





This is a [[7, 3, 1]] QEC code, which implements a <u>transversal</u> CCZ gate.

Finding a symmetric T-depth 1 circuit for binary AND



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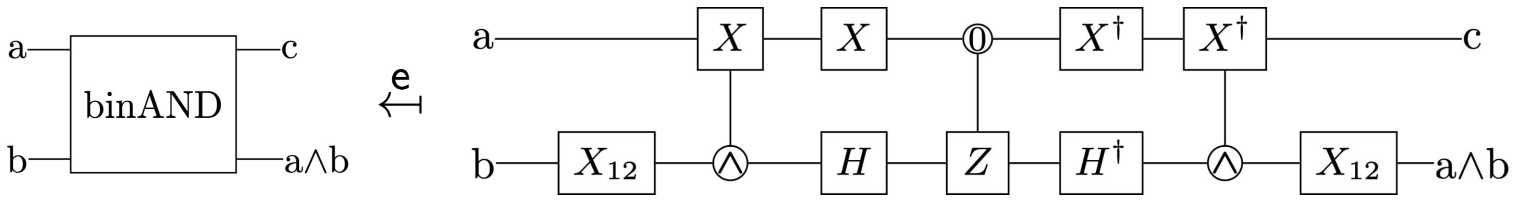
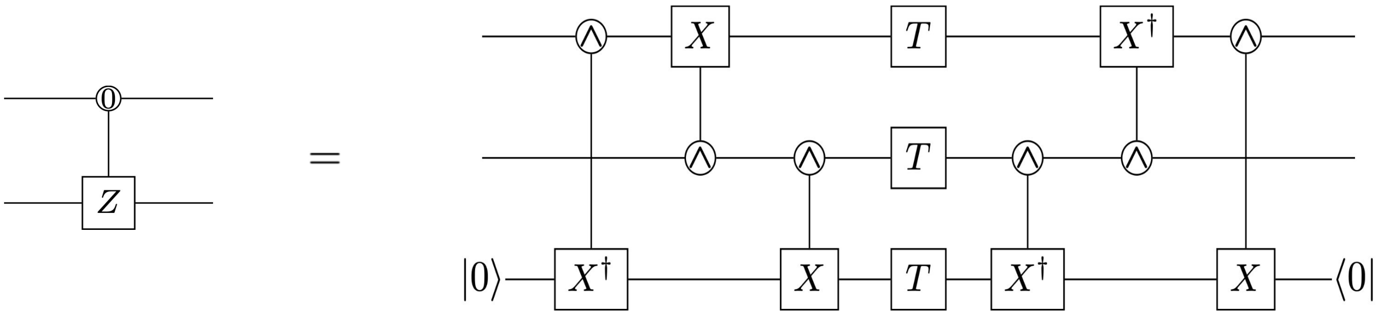
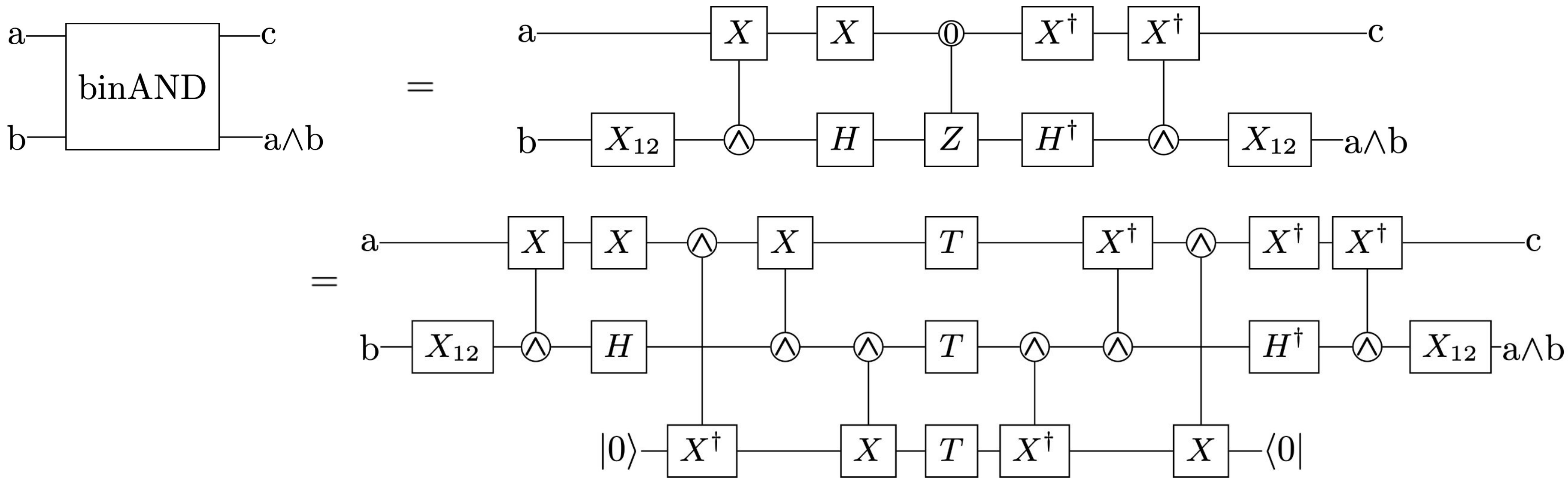




Circuit for 0-controlled Z from [BRS17]

Finding a symmetric T-depth 1 circuit for binary AND



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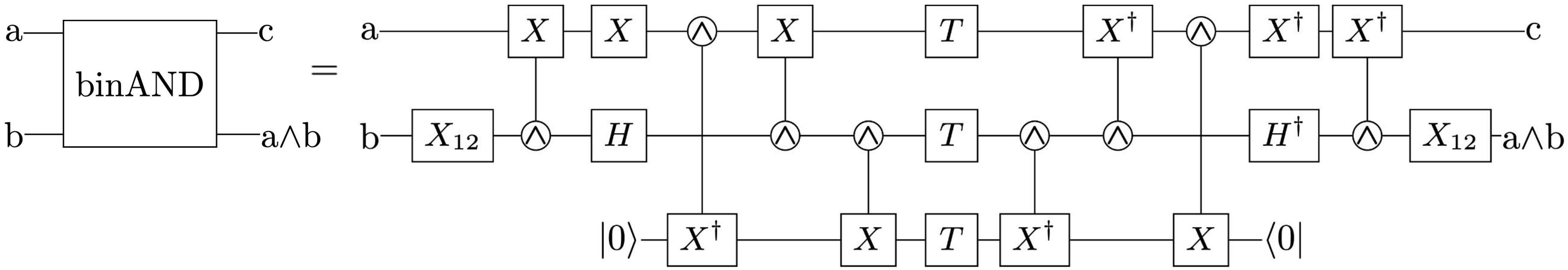
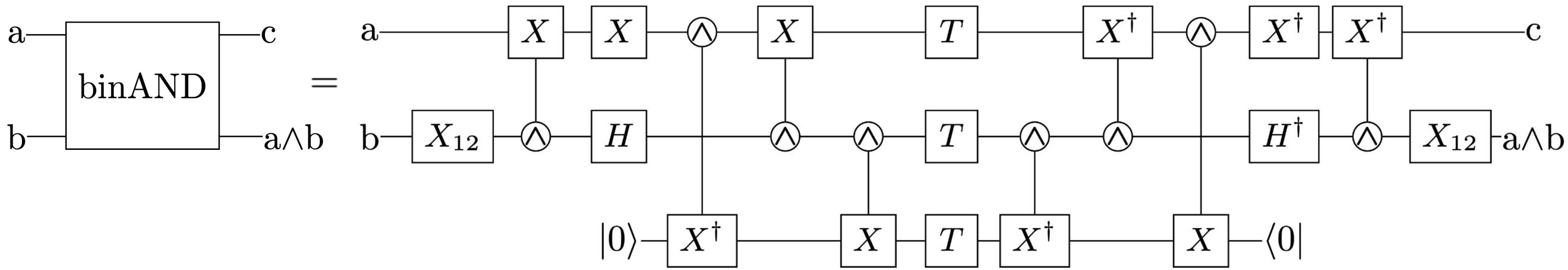
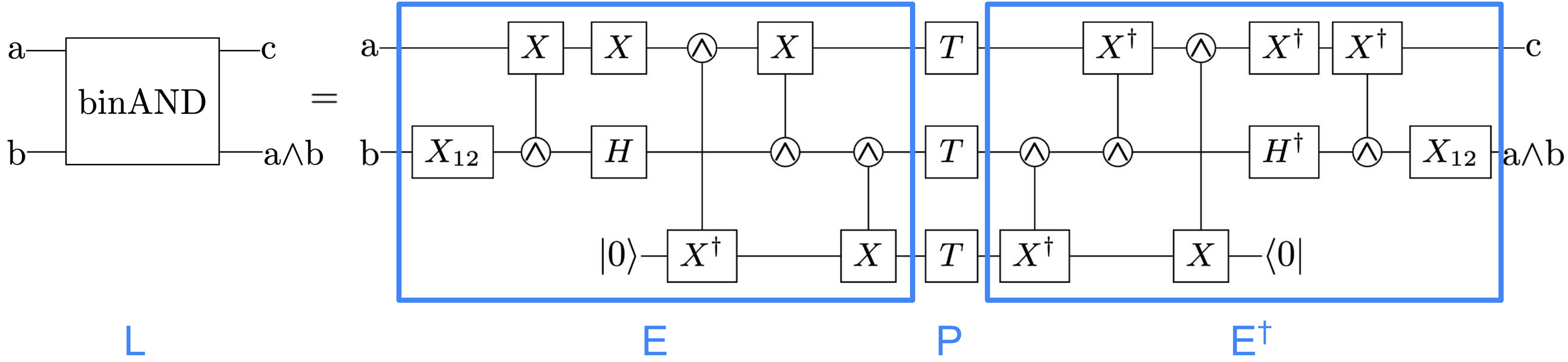
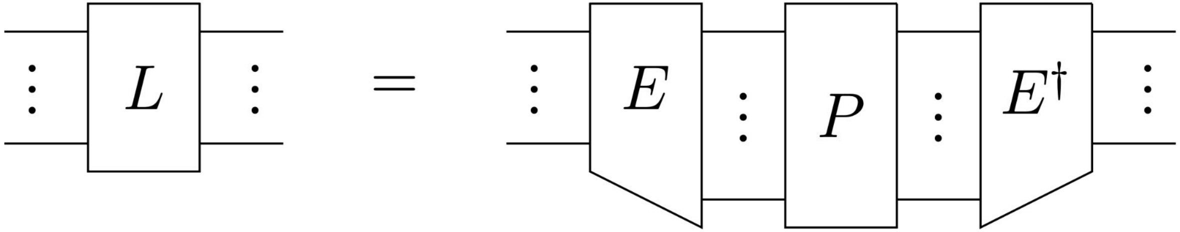


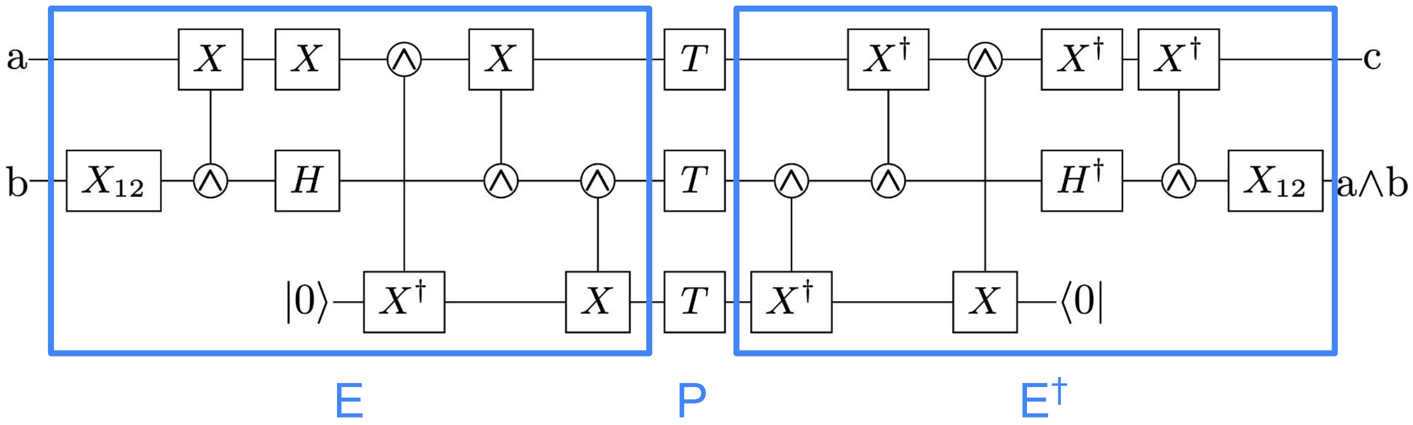
Finding a symmetric T-depth 1 circuit for binary AND



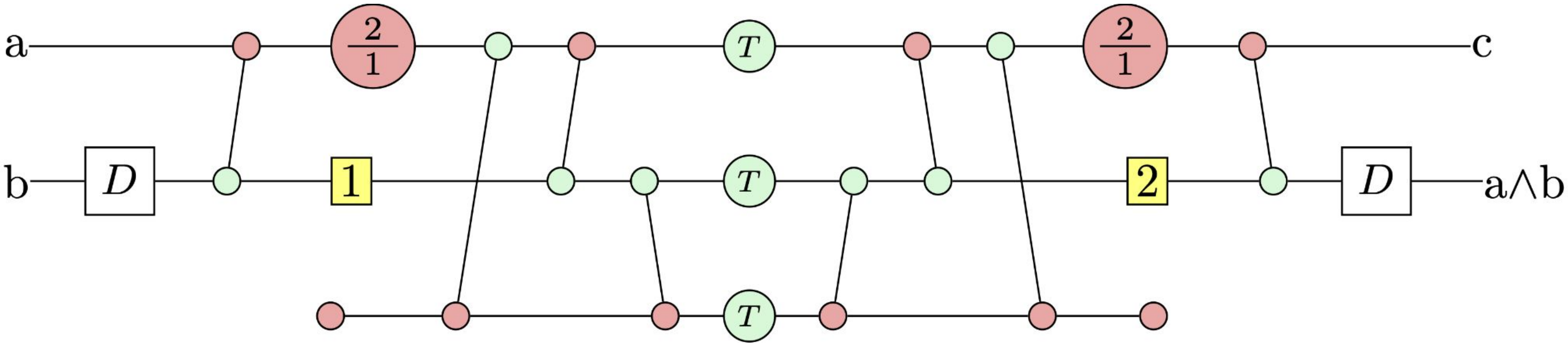
A symmetric T depth 1 circuit is a QEC code







In ZX-Calculus:

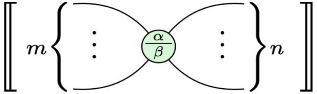


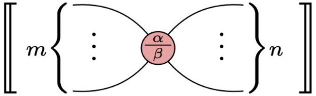
A very quick introduction to ZX calculus:

Any linear map on qubits/qutrits/qudits is representable as a <u>**ZX-calculus**</u> <u>**diagram**</u> made of Z and X spiders:

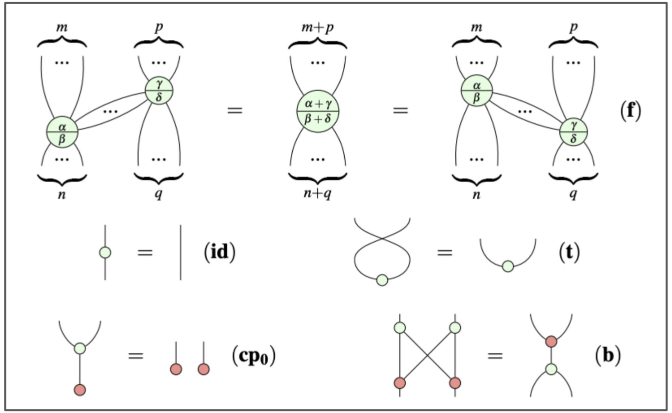
For qutrits:

$\omega = e^{\frac{2\pi i}{3}}$

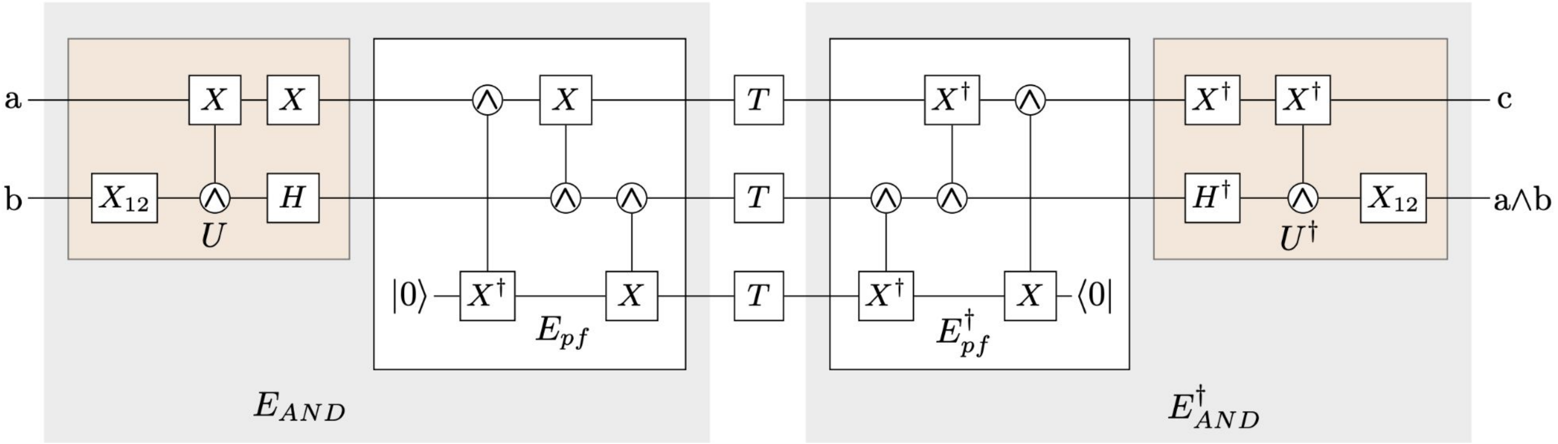
Z  $:= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + \omega^a |1\rangle^{\otimes m} \langle 1|^{\otimes n} + \omega^b |2\rangle^{\otimes m} \langle 2|^{\otimes n}$

X  $:= |+\rangle^{\otimes m} \langle +|^{\otimes n} + \omega^a |\omega\rangle^{\otimes m} \langle \omega|^{\otimes n} + \omega^b |\overline{\omega}\rangle^{\otimes m} \langle \overline{\omega}|^{\otimes n}$

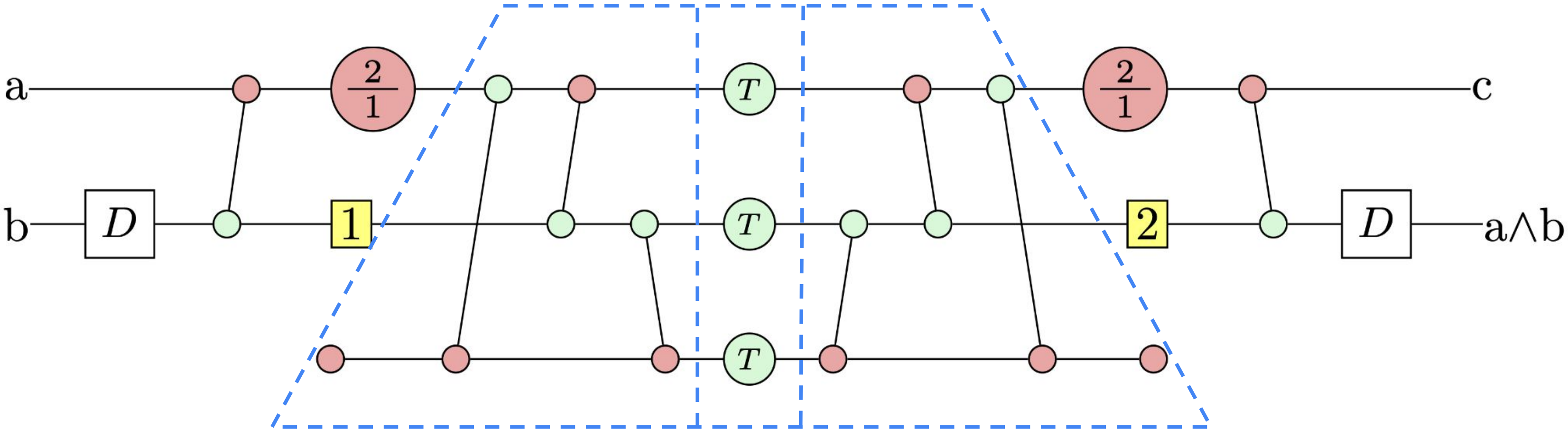
The qutrit ZX-calculus is <u>complete</u> for the stabilizer (Clifford) fragment of qutrit computation.



Example of some rewrite rules for the qutrit ZX calculus [TM21]



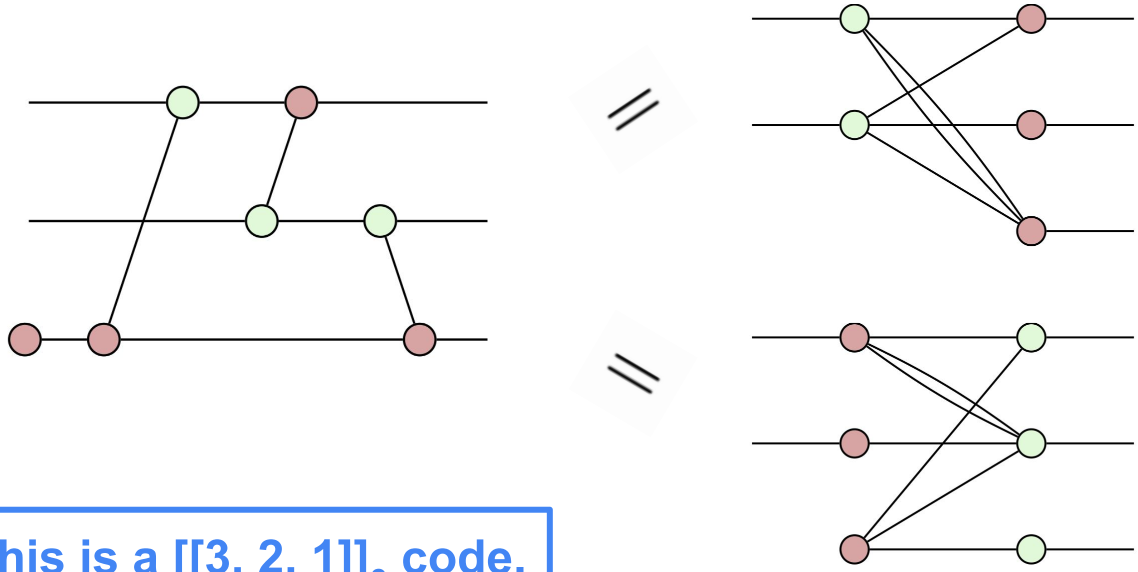
In ZX-Calculus:



Finding the code using ZX-calculus

Inner CX encoder circuit from the binary AND symmetric circuit:

Rewrite into normal forms for CSS codes in ZX:



$$\overline{X}_1 = X_1 X_3^2$$

$$\overline{X}_2 = X_1 X_2 X_3$$

No X-type stabilizers

$$\overline{Z}_1 = Z_1 Z_2^2$$

$$\boxed{\overline{Z}_2 = Z_2}$$

$$S_1^Z = Z_1 Z_2 Z_3$$

distance 1 :(

This is a [[3, 2, 1]]$_3$ code.

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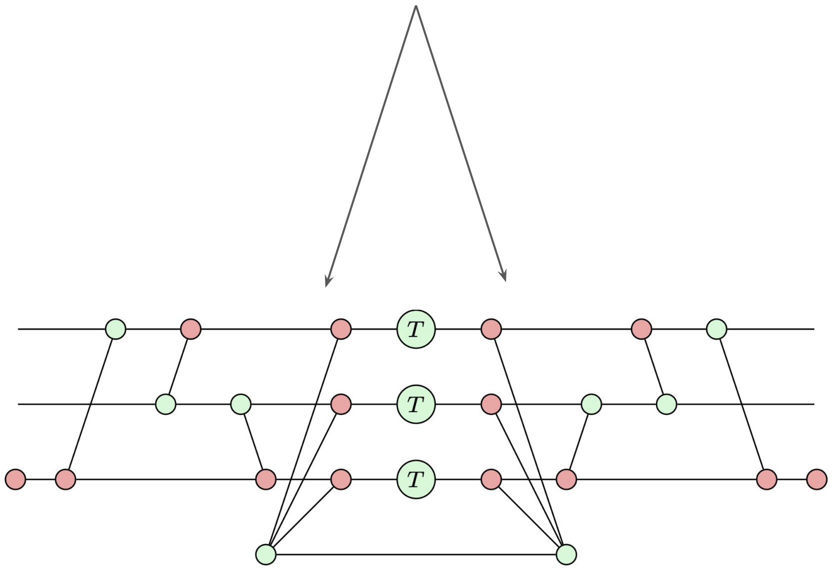
$[[3, 2, 1]]_3$ + weight 4 X-type stabilizer = $[[4, 2, 2]]_3$ code



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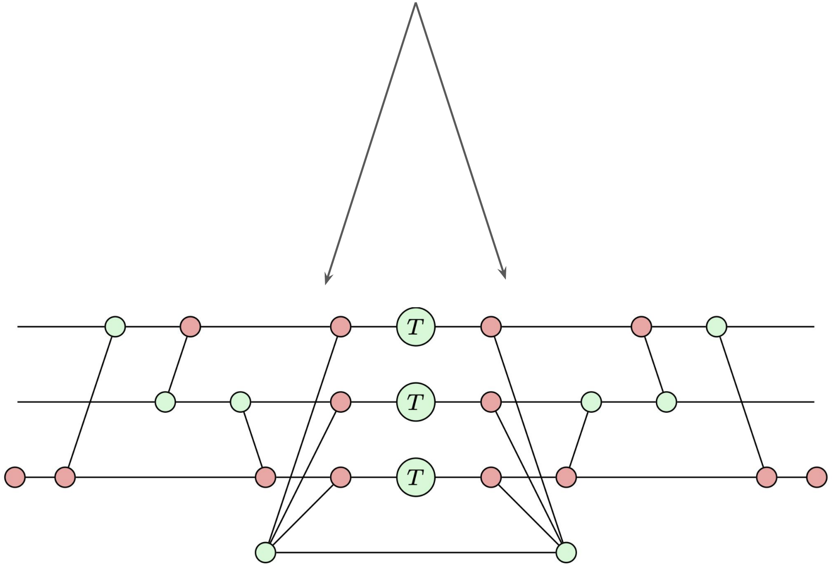
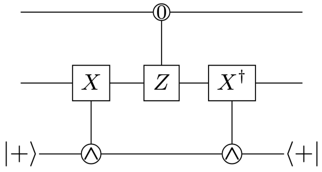
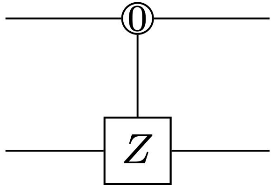


But this changes the logical!

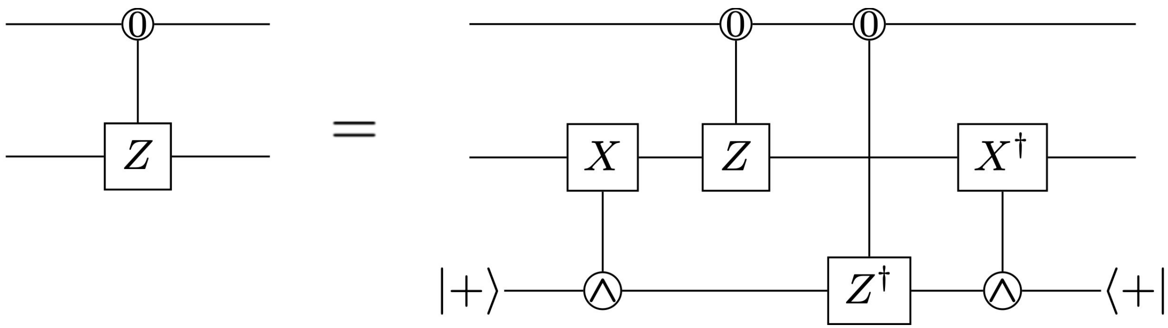
$=$



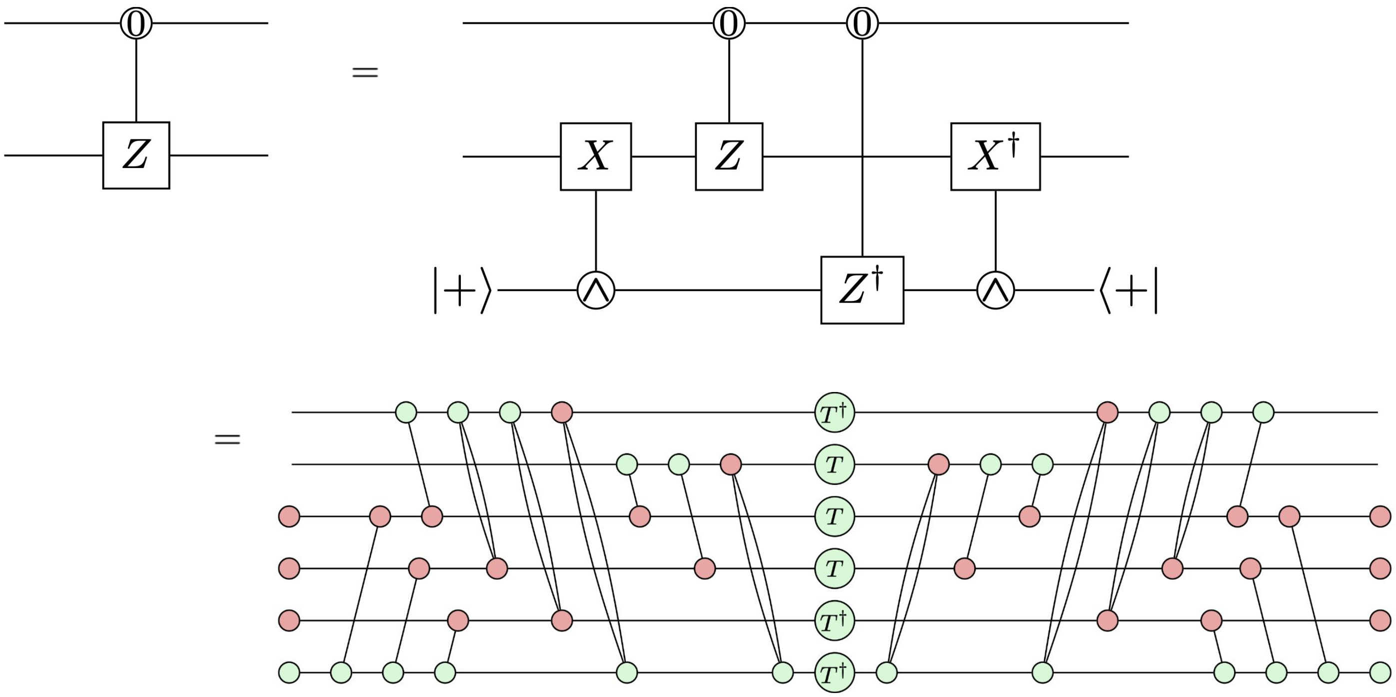
$\neq$



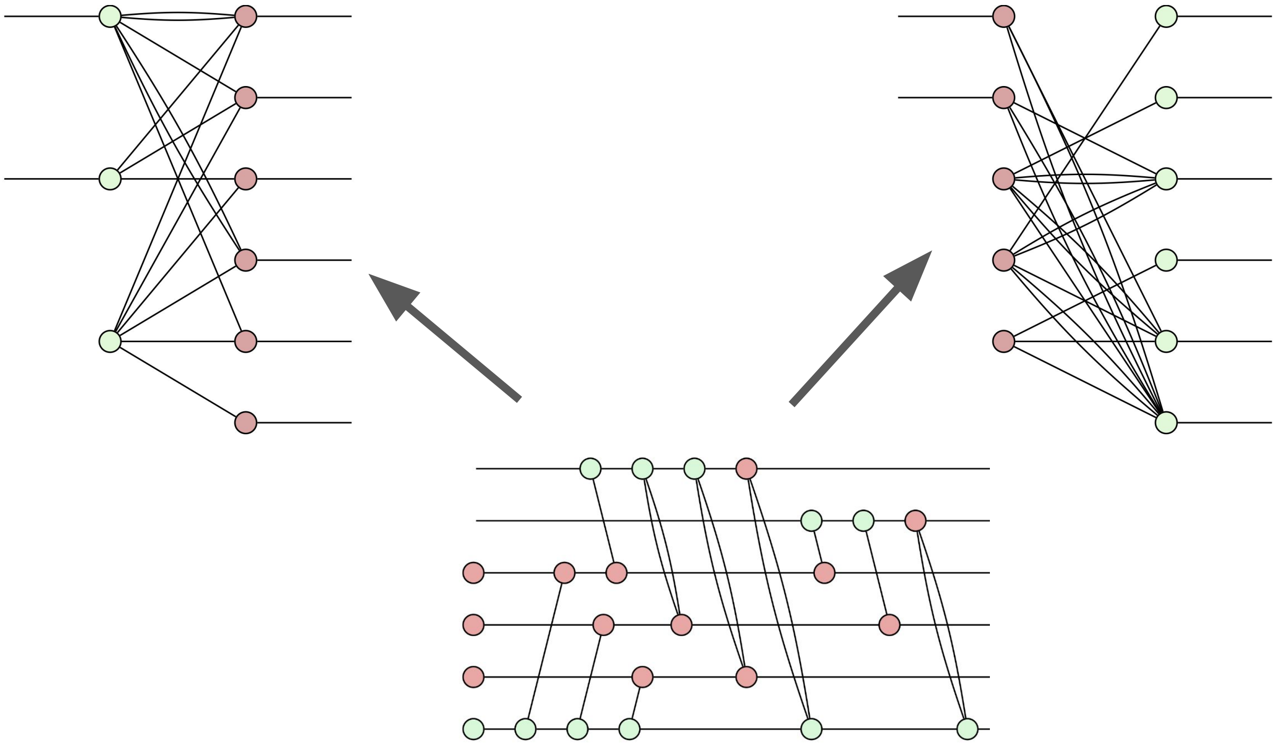
New circuit that preserves logical and increases distance



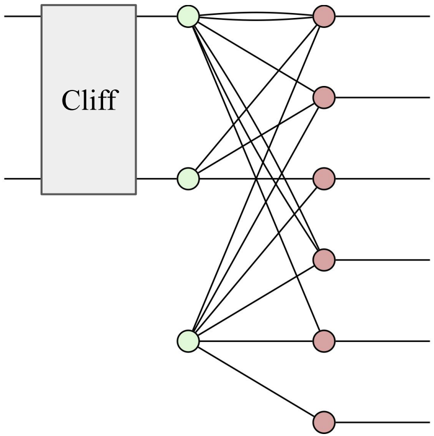
Rewriting as a symmetric T-depth 1 circuit



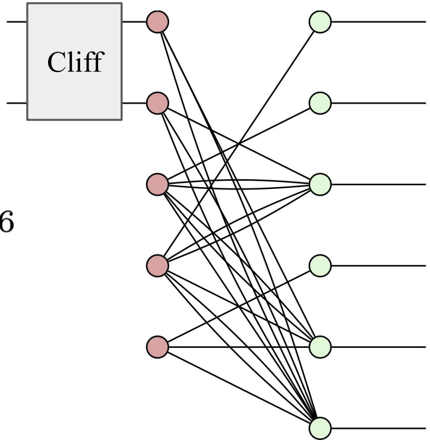
Finding the code’s stabilizers and logical operators



Qutrit codes with a transversal AND gate



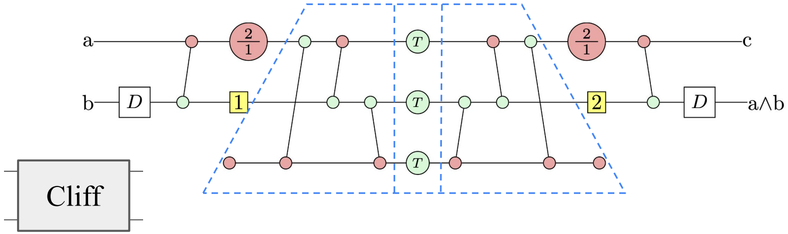
$$\overline{X}_1 = X_1^2 X_2 X_4^2 X_5$$
$$\overline{X}_2 = X_1 X_2^2 Z_3^2 X_4 X_5^2 Z_6$$
$$S_{X1} = X_1 X_2 X_3 X_4 X_5 X_6$$



$$\overline{Z}_1 = X_1 X_2 X_3 Z_4 Z_5^2$$
$$\overline{Z}_2 = X_1 X_2 X_3$$
$$S_{Z1} = Z_2 Z_3^2 Z_5^2 Z_6$$
$$S_{Z2} = Z_1 Z_3^2 Z_5 Z_6^2$$
$$S_{Z3} = Z_4 Z_5 Z_6$$

$d_x = 3$, $d_z = 2$

Recall the full AND circuit:



This is a $[[6, 2, 2]]_3$ code with a transversal AND gate.

Concatenate with the $[[8, 1, 2]]_3$ code from [CAB12] → $[[48, 2, 4]]_3$ code with transversal AND gate.

[CAB12] Campbell et al., Phys. Rev. X 2, 041021 (2012)

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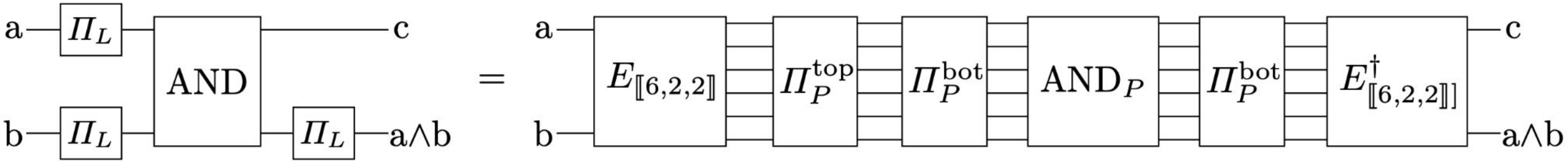
3. **Qubit subspace projection and magic state injection**

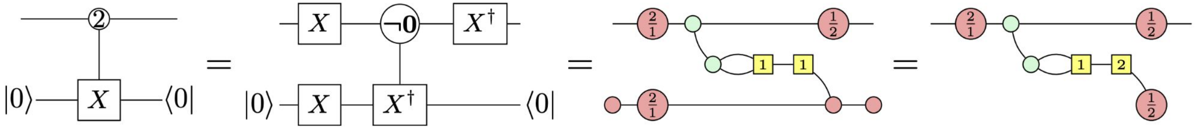
Qubit Subspace Projection

Idea: logical qubit computation using physical qutrits

- Encode physical qutrits into logical qutrits using a qutrit code
- **Project the logical qutrits into the logical qubit subspace**
- Logical qubits from physical qutrits (n = 6 physical qutrits → k = 2 logical qubits)



"qubit subspace projection" (SP) gadget:

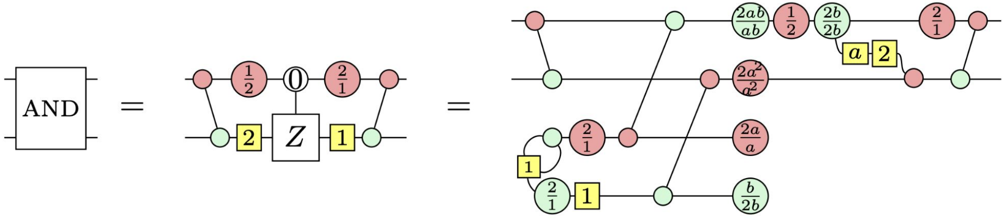


Magic state distillation and injection

How to implement an AND gate when computing with another quantum code that doesn't have a native transversal AND:

- Prepare resource ("magic") states for the AND gate, using distillation
- Execute AND gates fault-tolerantly in our working code, by injecting their magic states

AND injection circuit:



Key takeaways:

- Use qutrits to make binary AND “reversible”
- Construct quantum codes from a symmetric T depth-one circuit
- Result: quantum codes with a built-in transversal AND

Future directions:

- Further increase code distance
- Find other logical operators of these code
- Simulate threshold under depolarizing and leakage noise to compare with qubit codes
- Efficient implementations of other non-reversible binary gates on qudits

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