A Lean Mechanization of Reversible Occurrence Nets

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Introduction

- Goal: **formalizing reversible occurrence Petri nets**, and their properties in a proof assistant.
  - We focus on the reversible nets of Reversing Place Transition Nets (Melgratti, Mezzina, Ulidowski).
  - We prove several properties including the parabolic lemma.

Introduction

- Goal: **formalizing reversible occurrence Petri nets**, and their properties in a proof assistant.
  - We focus on the reversible nets of Reversing Place Transition Nets (Melgratti, Mezzina, Ulidowski).
  - We prove several properties including the parabolic lemma.
- We chose **Lean Prover** as proof assistant.
  - full programming language
  - interactive theorem prover

Net

Definitions

A **net** N is a tuple $(\alpha, \beta, {}^\bullet{-}_N, {-}^\bullet_N)$ where

- α is the set of **places**,
- β is the set of **transitions**, and
- ${}^\bullet{-}_N, {-}^\bullet_N : \beta \to \mathbb{N}^\alpha$ are functions that assign source and target to each transitions such that ${}^\bullet t \neq \emptyset$ and $t^\bullet \neq \emptyset$ for all $t \in \beta$.

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Net.lean

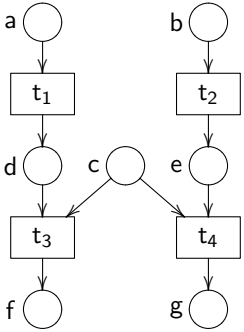
```
structure Net (α : Type) (β : Type) where
 pre: β → Multiset α
 post: β → Multiset α
 cons_prod: ∀ t, pre t ≠ ∅ ∧ post t ≠ ∅
```

Example

Net: $N = (\alpha, \beta, {}^\bullet{-}_N, {-}^\bullet_N)$

$$\alpha = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}, \mathsf{f}, \mathsf{g}\}$$

$$\beta = \{\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \mathsf{t}_4\}$$



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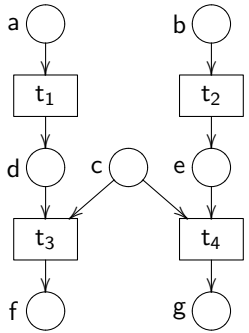

Net.lean

```
inductive Pl | a | b | c | d | e | f | g
inductive Tr | t₁ | t₂ | t₃ | t₄




.
```

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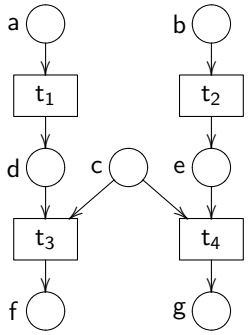

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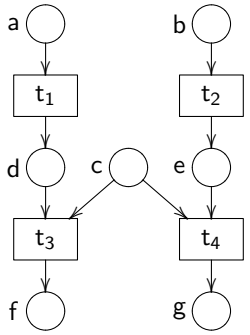

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def post₁ : Tr → Multiset Pl
 | t₁ => {d}
 | t₂ => {e}
 | t₃ => {f}
 | t₄ => {g}




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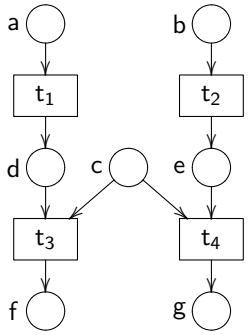

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def cons_prod₁ : ∀ t : Tr, pre ≠ ∅ ∧ post ≠ ∅ :=
  ...

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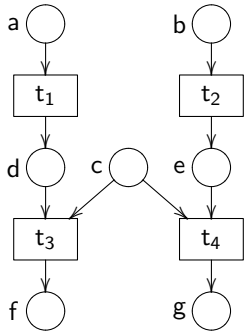

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def cons_prod₁ : ∀ t : Tr, pre ≠ ∅ ∧ post ≠ ∅ :=
  ...

def N1 : Net Pl Tr := ⟨pre₁, post₁, cons_prod₁⟩
```

Petri Nets

Definition

- A *marking* m of a net N is a multiset over the set of places, i.e., $m \in \mathbb{N}^{\alpha}$.
- A **Petri net** or **marked net** is a pair (N, m) where N is a net and m is a marking of N.

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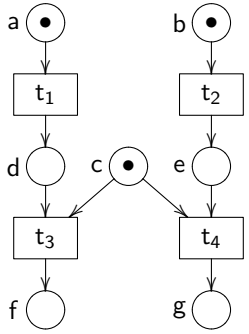
Net.lean

```
structure MarkedNet α β extends Net α β where
 m₀ : Multiset α
```

Example

Marked net: $(N, \mathsf{a} + \mathsf{b} + \mathsf{c})$



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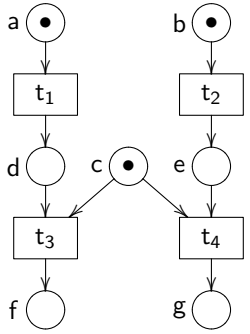

Net.lean

```
1 def M1 : MarkedNet Pl Tr := ⟨N1, {a, b, c}⟩
```

Nets

Preset and postset

Pre- and postsets for places

- **Preset:** ${}^\bullet a := \{t \mid a \in t^\bullet\}$
- **Postset:** $a^\bullet := \{t \mid a \in {}^\bullet t\}$

Nets

Preset and postset

Pre- and postsets for places

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Net.lean

```
def preset_p (N : Net α β) (p : α) : Set β :=
  {t | p ∈ N.post t}

def postset_p (N : Net α β) (p : α) : Set β :=
  {t | p ∈ N.pre t}

def preset_t (N : Net α β) (t : β) :=
  N.pre t

def postset_t (N : Net α β) (t : β):=
  N.post t
```

We introduce the shortland notation $\bullet\{\!\{N\}\!\}x$ and $x\bullet\{\!\{N\}\!\}$ for unified preset and postset syntax in Lean.

Firing

Definition

- A transition t is *enabled* at a marking m, denoted $m[\![\mathsf{t}]\!]$, if ${}^\bullet\mathsf{t} \leq m$.
- The *firing* of a enabled transition t at m is denoted by $m[\![\mathsf{t}\rangle m'$ if $m' = m - {}^\bullet\mathsf{t} + \mathsf{t}^\bullet$.

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Net.lean

```
def is_enabled (N : Net α β)
    (m : Multiset α) (t : β) : Prop :=
  •{N} t ≤ m

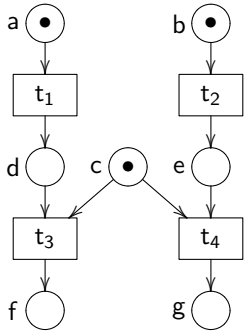
notation m "[[" t "]]{" N "}" =>
                        is_enabled ↑N m t

def marking_after_firing (N: Net α β)
    (m : Multiset α) (_: m[[t]]{N}) :=
  m - •{N} t + t•{N}
```

m[[e⟩{N}m' if firing t (enabled by e : m[[t]]{N}) produces m'.

Example

Firing: $(a + b + c)[\![t_1\rangle\!\rangle_{\{\!\{N\}\!\}}(d + b + c)$



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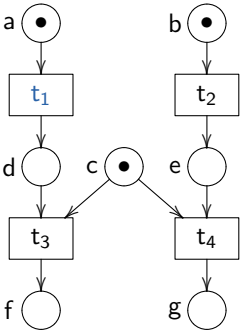


```
1  --Net N₁ was building previously
2
3  lemma et₁ : {a, b, c}⟦t₁⟧{{N₂}}:= by
4    unfold is_enabled preset_t N₂ Net.pre pre2
5    simp
6
7
8  .
```

Example

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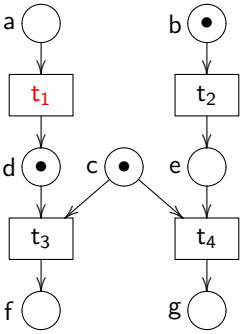


```
1  --Net N1 was building previously
2
3  lemma et1 : {a, b, c}⟦t1⟧{{N2}} := by
4    unfold is_enabled preset_t N2 Net.pre pre2
5    simp
6
7  example : {a,b,c} ⟦ et1⟩{{N1}} {d,b,c} := by
8    exact Multiset.add_comm ({b} + {c}) {d}
```

Semantics

Firing sequence

$s = \mathsf{t}_1; \mathsf{t}_2; \ldots; \mathsf{t}_n$ with *firing sequence*
$m[\![s\rangle\!\rangle m_n$ defined inductively by

- $m[\![\varepsilon\rangle m$ if $n = 0$
- $m[\![\mathsf{t}_1\rangle m_1 \wedge m_1[\![\mathsf{t}_2; \cdots; \mathsf{t}_n\rangle\!\rangle m_n$ if $n > 0$

$s = \varepsilon$ is the empty sequence.

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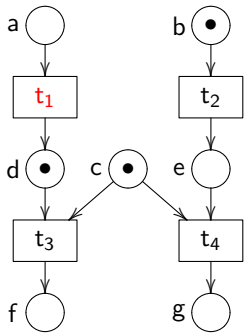
Nets.lean

```
inductive firing_sequence : Net α β →
    Multiset α → List β → Multiset α → Prop
| empty m :
    firing_sequence _ m [] m
| step {N : Net α β} {t} {ts}
    (et : m [[t]]{{N}})
    (f : m [[et⟩{{N}} m')
    (fs : firing_sequence N m' ts m''):
    firing_sequence N m (t :: ts) m''
```

`m[[s⟩⟩{{N}}m'` if the firing sequence from `m` through the sequence `s` is produces `m'`.

Example

Firing sequence: $(a + b + c)[\![t_1; t_2; t_3\rangle\!\rangle_{\{\!\{N\}\!\}}(e + f)$



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example : {a,b,c} ⟦ [t₁, t₂, t₃]⟩⟩ {{N₂}} {e,f} := by
  obtain et₁ : {a, b, c}⟦t₁⟧{{N₂}}:= ...
  obtain fir₁ : {a, b, c} ⟦et₁⟩{{N₂}} {d, b, c} := ...
  sorry

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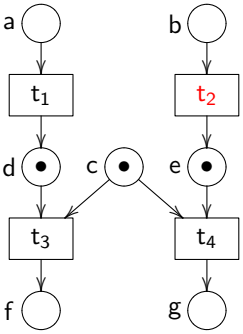

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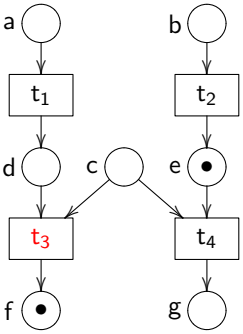

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  obtain et₃ : {d,c,e}⟦t₃⟧{{N₂}} := ...
  obtain fir₃ : {d,c,e}⟦et₃⟩{{N₂}} {e,f} := ...
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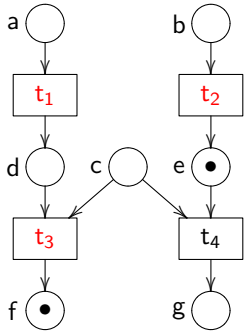

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  apply firing_sequence.step et₁ fir₁ <|
    .step et₂ fir₂ <|
    .step et₃ fir₃ <|
    .empty {e,f}
```

Occurrence nets

Causal dependency

Causal Dependency

The *immediate dependency relation* $<_{\{\!|N|\!\}} \subseteq (\alpha \cup \beta) \times (\alpha \cup \beta)$ is defined by

$$\{(x, y) \mid (y \in \beta \wedge x \in {}^\bullet y) \vee (x \in \beta \wedge y \in x^\bullet)\}$$

- Causal relation: transitive closure

$$\prec_{\{\!|N|\!\}} := <^*_{\{\!|N|\!\}}$$

- Causal preorder: reflexive-transitive closure

$$\preceq_{\{\!|N|\!\}} := \prec_{\{\!|N|\!\}} \cup \{(x, x) \mid x \in \alpha \cup \beta\}$$

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Occurrence.lean

```
1  def immediate (N: Net α β) :
       (α ⊕ β) → (α ⊕ β) → Prop
2    | inl p, inr t  =>  p ∈ •{{N}} t
3    | inr t, inl p  =>  p ∈ t •{{N}}
4    | _, _          =>  false
5
6  notation l "<{{"N"}}" r => immediate N l r
7
8
9
10
11
12
13
14
15
16
17
18 .
```

Occurrence nets

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  | _, _          =>  false

notation l "<{{" N "}}" r => immediate N l r

def causal (N : Net α β) :
    (α ⊕ β) → (α ⊕ β) → Prop :=
  TransGen (immediate N)

notation l "≺{{" N "}}" r => causal ↑N l r

def preorder (N : Net α β) :
    (α ⊕ β) → (α ⊕ β) → Prop :=
  ReflTransGen (immediate N)

notation l "≼{{" N "}}" r => preorder ↑N l r
```

Occurrence net

Conflict

- Two different transitions t and t' are in *immediate conflict* if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$, we denote $t \#_0 t$.
- $x, y \in \alpha \cup \beta$ are in *conflict* if there are transitions t, t' such that: $t \preceq x$, $t' \preceq y$ and $t \#_0 t'$.

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Occurrence.lean

```
def immediate_conflict (N : Net α β) (t t' : β) : Prop :=
  t ≠ t' ∧ ¬ (Disjoint (•{N} t) (•{N} t'))

notation l "#₀{" N "}" r => immediate_conflict ↑N l r

def conflict (N : Net α β) (x y : α ⊕ β) : Prop :=
  ∃ t t' : β, inr t ≼{N} x ∧ inr t' ≼{N} y ∧ t #₀{N} t'

notation l "#{" N "}" r => conflict ↑N l r
```

Occurrence net

We recall the definition of occurrence nets

Occurrence net

A marked net (N, m_0) is an *occurrence net* if it satisfies the following statements:

1. N is acyclic;
2. there is no self-conflict, i.e., $\neg(\mathrm{t} \#_{\{\!\{N}\!\}} \mathrm{t})$ for all transition t;
3. there is no backward conflict;
4. N is 1-safe, i.e., marking is a set;
5. the initial marking is the set of initial places, i.e., m_0 is minimal.

Occurrence net

We recall the definition of occurrence nets

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Occurrence

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4. for every transition t, •t and t• are sets.
5. the initial marking is the set of initial places, i.e., m_0 is minimal.

Occurrence.lean

```
structure is_occurrence (N : Net α β) where
  acyclicity : acyclic N
  no_self_conflict :
    ∀ t : β, ¬inr t #{|N|} inr t
  no_backward_conflict :
    ∀ a : α, ¬backward_conflict N a
  set_preset :
    ∀ t : β,  Nodup (•{|N|} t)
  set_postset :
    ∀ t : β, Nodup (t •{|N|})

structure is_marked_occurrence (M :
    MarkedNet α β) : Prop where
  occurrence : is_occurrence M.toNet
  initial_set : Nodup M.m₀
  inital_minimal : ∀ {x}, x ∈ M.m₀ → x ∈
      minimal M.toNet
```

Occurrence

Occurrence.lean

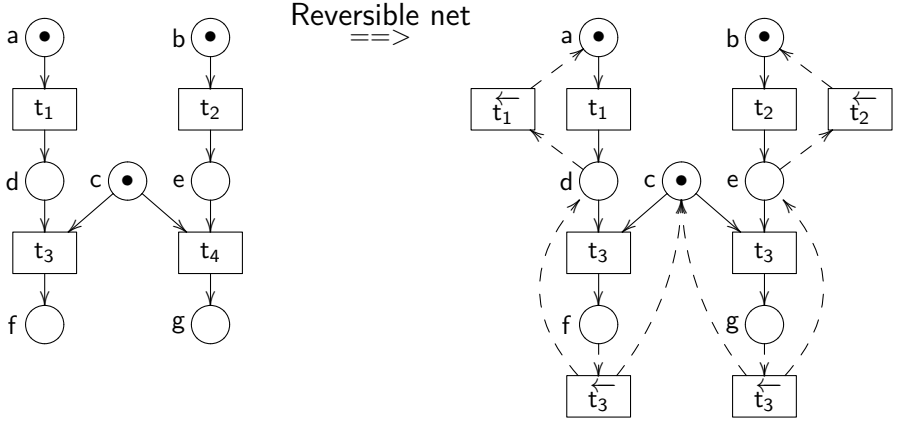
```
/-- This def. is in Nets.lean file -/
def safe (M : MarkedNet α β) : Prop :=
  ∀ m, M ⇝ m → Nodup m

theorem occ_net_safe (MO : is_marked_occurrence M) : safe M
```

M ⇝ m is true if m ∈ reach(M).

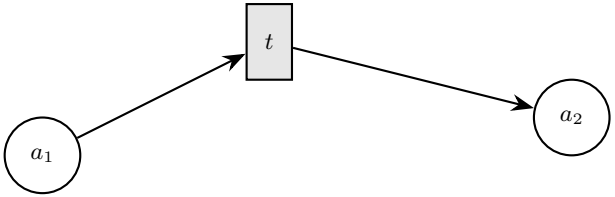
Reversing nets

How do you obtain a reversible net?



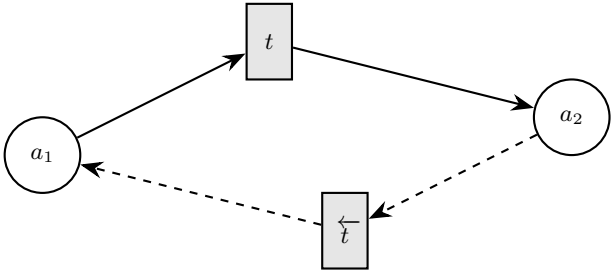
Reversing nets

Transition Type



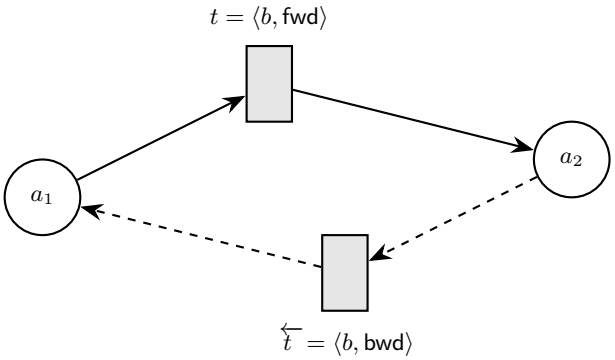
Reversing nets

Transition Type



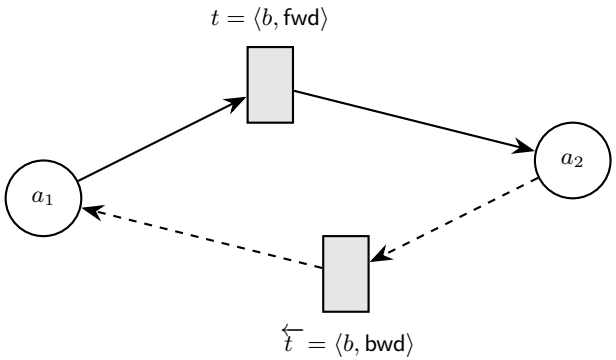
Reversing nets

Transition Type



Reversing nets

Transition Type



ReversibleOccurrence.lean

```
1  inductive TransitionType
2  | fwd : TransitionType
3  | bwd : TransitionType
4
5  abbrev Transition (β : Type) :=
       β×TransitionType
```

Reversing nets

Definition

A *reversible occurrence net* is a net $N = (\alpha, \beta \uplus \{ \overleftarrow{t} \mid t \in \beta \}, {}^\bullet\_, \_^\bullet)$ defined by the following conditions:

1. $(\alpha, \beta, ({}^\bullet\_)_{|\beta}, (\_^\bullet)_{|\beta})$ is an occurrence net, where $f_{|\beta}$ denotes the restriction of the function f to the domain β.
2. $\forall t\ \exists t',\ {}^\bullet t = t'^\bullet \wedge t^\bullet = {}^\bullet t'$.

Reversing nets

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ReversibleOccurrence.lean

```
structure Reversible α β
    extends Net α (Transition β) where
  welldef (t t' : Transition β) :
    t ⤚ t' → •{{toNet}}t = t'•{{toNet}}

structure ReversibleOccurrence α β extends
    Reversible α β where
  isOccurrence :
    is_occurrence (fwd_subnet toNet)

structure MarkedReversibleOccurrence α β
    extends Reversible α β where
  m₀ : Multiset α
  isFwdMarkedOccurrence :
    is_marked_occurrence ⟨fwd_subnet toNet, m₀⟩
```

- `fwd_subnet`: is the function that take a net of type `N:α (β×TransitionType)` and return their forward net.

Trace Equivalence

Equivalence on firing sequences:
the smallest congruence $\asymp$ on transition sequences closed under the following rules:

(i) $t; t' \asymp t'; t$ if $t \; co \; t'$,

(ii) $t; \overleftarrow{t} \asymp \varepsilon$, and

(iii) $\overleftarrow{t}; t \asymp \varepsilon$.

Trace Equivalence

The *equivalent firing sequences*, is the smallest congruence $\asymp$ on transition sequences closed under the following rules:

(i) $t; t' \asymp t'; t$ if $t \; co \; t'$,

(ii) $t; \overleftarrow{t} \asymp \varepsilon$, and

(iii) $\overleftarrow{t}; t \asymp \varepsilon$.

Equivalence.lean

```
inductive TrEq (R : Reversible α β) :
  List (Transition β) → List (Transition β) → Prop
| swap : (t t' : Transition β) →
  Disjoint (t•{|R|}) (•{|R|} t') → Disjoint (t'•{|R|}) (•{|R|} t) →
  TrEq R [t, t'] [t', t]
| canc_l : (t t' : Transition β) → t ↚ t' →
  TrEq R [t, t'] []
| canc_r : (t t' : Transition β) → t ↚ t' →
  TrEq R [] [t, t']
...
notation:50 s₁:51 "≍{|" R "|} " s₂:51 => TrEq R s₁ s₂

/--We omit the proofs that ≍ is an equivalence relation.-/

lemma TrEq_congruence (h : s₁ ≍{|R|} s₂) (h' : s₃ ≍{|R|} s₄) :
    (s₁ ++ s₃) ≍{|R|} (s₂ ++ s₄)
```

Parabolic lemma

Formal statement

Lemma (Parabolic)

Let R be a reversible occurrence net, m a reachable marking, and $m[\![\omega\rangle\!\rangle_{\{\!\{R\}\!\}} m'$ a firing sequence. Then there exist two sequences of transitions, ω' and ω'', such that ω' consists only of backward transitions, ω'' consists only of forward transitions, $\omega \asymp_{\{\!\{R\}\!\}} \omega';\omega''$, and $m[\![\omega';\omega''\rangle\!\rangle_{\{\!\{R\}\!\}} m'$.

Parabolic lemma

Parabolic.lean

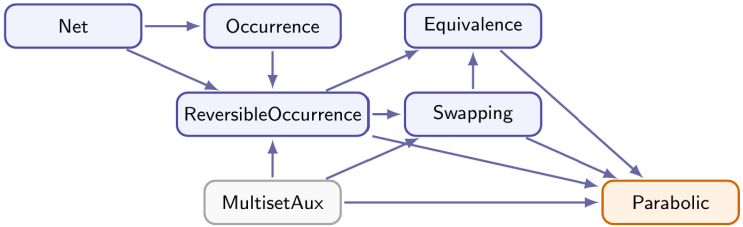
```
variable {RO : ReversibleOccurrence α β}

def parabolic_decomp (s s' s'' : List (Transition β)) (m m' : Multiset α) :=
  ⇜s' ∧ s'' ⇝ ∧ m [[s' ++ s'']]{{RO}} m' ∧ s ≍ {{RO.toReversible}} (s' ++ s'')

theorem parabolic (MO : MarkedReversibleOccurrence α β)
    (rr : ⟨MO.toNet, MO.m₀⟩ ⇝ m) (fs : m [[s]]{{MO}} m') :
    ∃ s' s'', parabolic_decomp (RO := marked_RO_to_RO MO) s s' s'' m m'
  induction fs with
  | @empty m =>
    exists [], []; simp_all [parabolic_decomp, all_backward, all_forward]
    exact ⟨.empty m, TrEq_refl []⟩
  | @step t fs' m m₁ _ et f _ IH =>
    ...
    by_cases h : isBwd t
    · ... /-- Case h : ◁t -/
    · ... /-- Case h : ¬◁ t -/
```

Overview of the formalization

This library contains the fully formalized of the theory of reversible Petri nets with causally consistent reversibility. The main folder (`PetriNet`) is organized as follow files `.lean`:



Lines of code	def	lemma	theorem
>2.8 k	62	136	14

Overview of the formalization

Blueprint and documentation

- Open repository available at: `github.com/danieldavalos93/reversible-occ-nets`.
  - **Lean version:** 4.30.0
  - **Mathlib version:** v4.30.0
- Blueprint:
  - website
  - pdf
- Documentation:
  - website
- Dependency graph

Conclusion and future works

Our contribution:

- First mechanization of reversible Petri nets on Lean Prover.
- We mechanize key results, including the backtracking, loop and parabolic lemmas.

Scenarios for future extensions:

- Colored Petri net.
- Probabilistic models for reversible nets.
- Reversible and concurrent models of quantum computing.

Thank you!

Trace Equivalence

Disjoint implies concurrence

$$
\begin{aligned}
t^\bullet \cap {}^\bullet t' = \emptyset &\Rightarrow {}^\bullet \overleftarrow{t} \cap {}^\bullet t' = \emptyset \\
&\Rightarrow \exists\, \overleftarrow{t}, t'.\ t \preceq \overleftarrow{t} \wedge t' \preceq t' \wedge \neg t \#_0 t' \\
&\Rightarrow \neg(t \# t') \\
&\Rightarrow \neg(t \preceq t') \wedge \neg(t' \preceq t) \wedge \neg(t \# t') \equiv t \; co \; t'
\end{aligned}
$$

Reversing nets

Reverse and Inverse definitions

- $\text{opposite}(tt) = \begin{cases} \text{bwd if } tt = \text{fwd} \\ \text{fwd if } t_t = \text{bwd} \end{cases}$
- Reverse: $\overleftarrow{\overleftarrow{t}} = (\pi_1(t), \text{opposite}(\pi_2(t)))$
- Inverse: $t \leftarrow\!\!\rightarrow t'$ if and only if $\overleftarrow{\overleftarrow{t}} = t'$

ReversibleOccurrence.lean

```
1  def opposite (tt : TransitionType)
2      : TransitionType :=
3    match tt with
4    | fwd  => bwd
5    | bwd  => fwd
6
7  def reverse (t : Transition β) : Transition β :=
8    (t.fst, opposite (t.snd))
9
10 prefix:300 "←" => reverse
11
12 def inverse (t t' : Transition β) : Prop :=
13   t = ←t'
14
15 notation:300 l:300 "←→" r:301 => inverse l r
```

Backward conflict

$a \in \alpha$ is in backward conflict over a net N if there are two different transitions t, t' such that $t < a$ and $t' < a$.

Parabolic lemma

Swapping lemma instead square lemma

Swapping.lean

```
def no_rev_in_list (t : Transition β) (ls : List (Transition β)) :=
  ∀ t' ∈ ls, ¬(t ⟵ t')

infix:50 " ∉ʳ " => no_rev_in_list

lemma swapping [DecidableEq α] (MO : MarkedReversibleOccurrence α β)
    (rr : ⟨MO.toNet, MO.m₀⟩ ⇝ m)
    (bw : ◁◁s) (ft : ▷t) (nrev : t ∉ʳ s)
    (fs : m ⟦[t] ++ s⟩⟪MO⟫ m') :  m ⟦s ++ [t]⟩⟪MO⟫ m' := ...
```

- ▷t: t is a forward transition
- ◁◁ s: s is a list containing only backward transitions