Exact Quantum State Preparation with the Standard Recursive Block Basis

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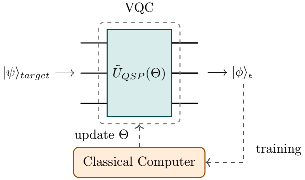
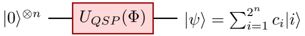
Preamble: QSP Circuit Design

- **Variational** Approaches
  - low-depth circuit ansatz $\tilde{U}_{QSP}(\Theta)$
  - non-convex optimization
  - local minima, barren plateaus
- **Exact** Synthesis Methods
  - circuit for entire $U_{QSP}(\Phi)$
  - exponential growth of parameters
  - rapidly increasing circuit depth





QSP traditional framework + **SRBB** ⇒ transition "*variational to exact*"

- no increase in depth/gate count compared to V-SRBB
- no optimization-induced limitations
- same expressive structure (complete basis for $\mathfrak{su}(2^n)$)

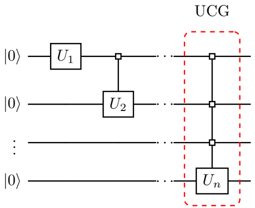
Summary

Gate Synthesis Tools

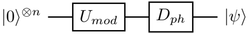
Uniformly Controlled Gate (UCG)

$$\boxed{\mathrm{F}^{\mathrm{n}}_{\mathrm{n}-1}[\mathrm{U}(2)] = \bigoplus_{\mathrm{j}=1}^{2^{\mathrm{n}-1}} \mathrm{U}_{\mathrm{j}}}$$

$$\mathrm{U}_{\mathrm{j}} = \mathrm{e}^{\mathrm{i}\alpha_{\mathrm{j}}} \mathrm{R}_{\mathrm{z}}(\beta_{\mathrm{j}}) \mathrm{R}_{\mathrm{y}}(\gamma_{\mathrm{j}}) \mathrm{R}_{\mathrm{z}}(\delta_{\mathrm{j}})$$



Historical Framework for $\mathrm{U}_{\mathrm{j}} \in \mathrm{SU}(2)$



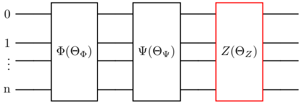
$\boxed{\text{UCG synthesis + BST diagram}}$

Standard Recursive Block Basis (SRBB)

$$(\theta_1, \ldots, \theta_{4^{\mathrm{n}}}) \in \mathbb{R}^{4^{\mathrm{n}}} \xrightarrow{\text{exp}} \prod_{\mathrm{j}=1}^{4^{\mathrm{n}}} \mathrm{e}^{\mathrm{i}\,\theta_{\mathrm{j}} \mathbf{B}_{\mathbf{j}}} \in \mathrm{U}(2^{\mathrm{n}})$$

$$\boxed{\mathcal{U}^{SRBB}_{\text{approx}} \equiv \prod_l Z_l(\Theta_Z)\Psi_l(\Theta_\Psi)\Phi_l(\Theta_\Phi)}$$

[R.S.Sarkar et al., arXiv:2405.00012]



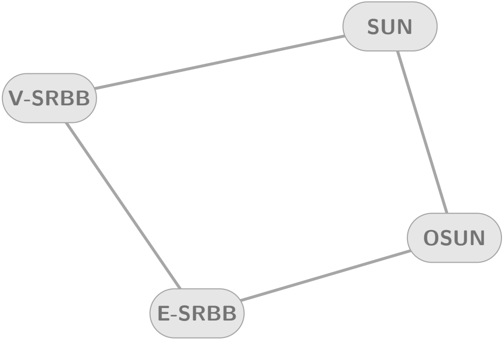
$$\mathrm{Z}(\Theta_{\mathrm{Z}}) = \prod_{\mathrm{j}=2}^{2^{\mathrm{n}}} \mathrm{e}^{\mathrm{i}\left(\theta_{\mathrm{j}^2-1} \mathrm{B}_{\mathrm{j}^2-1}\right)}$$

$$\text{CNOT}_Z,\ \text{Depth}_Z \sim \Theta(2^n)$$

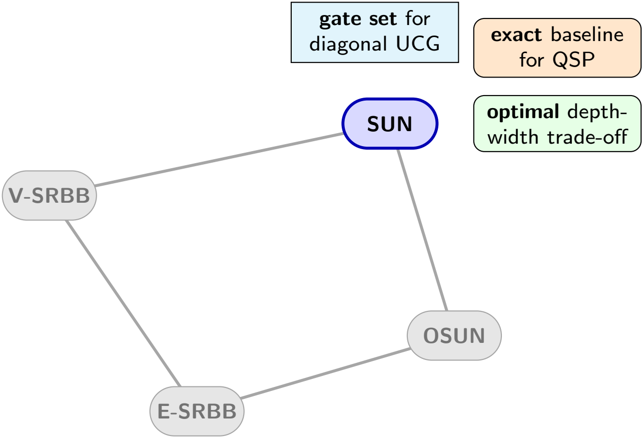
- single-layer Lie-based QNN
- CNOT-optimization

Paper: Quantum 10 (2026)
GitHub: qslab-unipr/srbb-synthesis

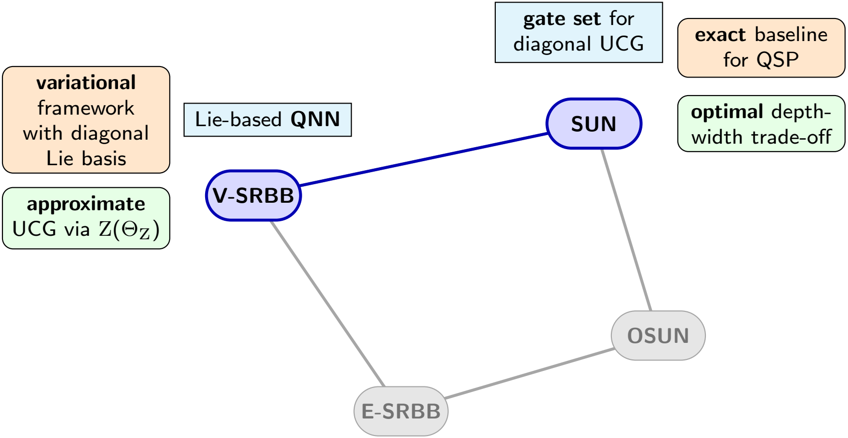
Overview



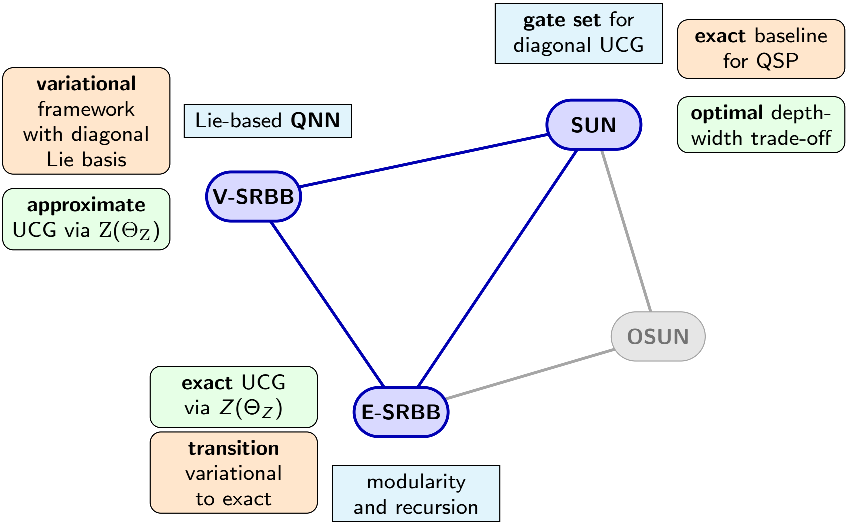
Overview


gate set for diagonal UCG
exact baseline for QSP
SUN
optimal depth-width trade-off
V-SRBB
OSUN
E-SRBB

Overview


gate set for diagonal UCG
exact baseline for QSP
variational framework with diagonal Lie basis
Lie-based **QNN**
SUN
optimal depth-width trade-off
V-SRBB
approximate UCG via $\mathrm{Z}(\Theta_{\mathrm{Z}})$
OSUN
E-SRBB

Overview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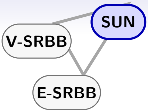
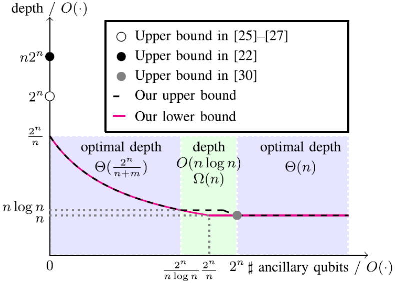
exact UCG via $Z(\Theta_Z)$
E-SRBB
transition variational to exact
modularity and recursion

Exact QSP with Ancillae: SUN





[X.Sun et al., arXiv:2108.06150]

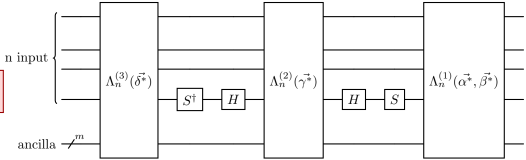
- **Optimal** width-depth bounds
- Λ-operator in **5 stages**
- ancillae $m \in [2n, \frac{2^n}{n}]$
  $D_\Lambda = \mathcal{O}(\log_2 m + \frac{2^n}{m})$
  $W_\Lambda = \mathcal{O}(2^n)$

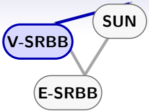
Paper: Springer RC25
GitHub: qslab-unipr/qsp-sun

$\mathrm{U}(2)$-block UCG decomposition

$\mathrm{R_y}(\gamma_j) \equiv \mathrm{SH} \cdot \mathrm{R_z}(\gamma_j) \cdot \mathrm{HS}^\dagger$

$\mathrm{UCG} \rightarrow 3 \cdot \mathrm{UCR_z} \rightarrow 3\Lambda$

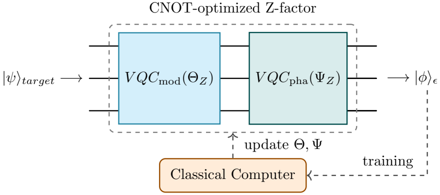




Approximate QSP: V-SRBB

Quantum Neural Network

designed on diagonal SRBB



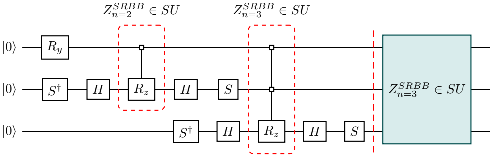
GitHub:
qslab-unipr/aquaman-sp

- **Diagonal** $\mathrm{SU}(2^n)$ generators
- **Modularity** by Z-factor
- $D_{\mathrm{QSP}}(n) = 6 \cdot 2^n - \frac{n^2+7n}{2} - 3$
- $N_{\mathrm{rot}}(n) = 3 \cdot 2^n - n - 3$
- $N_{\mathrm{CNOT}}(n) = 3 \cdot 2^n - 2n - 4$

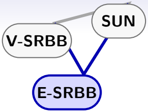
gate set change

$$\mathrm{R_y}(\gamma_j) \equiv \mathrm{SH} \cdot \mathrm{R_z}(\gamma_j) \cdot \mathrm{HS}^\dagger$$

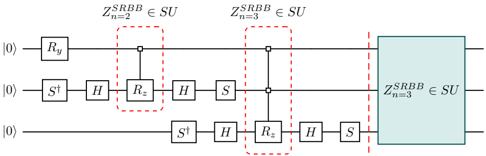
$$\boxed{\mathrm{UCG} \rightarrow 1 \cdot \mathrm{UCR_z} \rightarrow \mathrm{Z}_n^{SRBB}}$$



From QNN to Exact QSP: E-SRBB



Circuit for SRBB Insertion



Lie Map Inversion

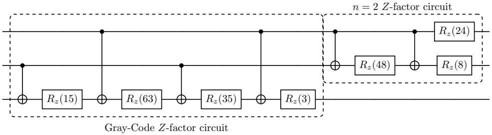
$$\boxed{M_k \vec{x}^{(k)}_{mod} = \vec{b}^{(k)}_{mod}}$$

$$\boxed{M_n \vec{x}^{(n)}_{phase} = \vec{b}^{(n)}_{phase}}$$

First SRBB-driven exact synthesis of diagonal UCG

Gray $n=2$		Element	Gray $n=3$			Element
			0	0	0	-
			0	**0**	**1**	$\boldsymbol{U_3}$
0	0	-	**0**	**1**	**1**	$\boldsymbol{U_{15}}$
0	1	U_3	0	1	0	U_8
1	1	U_{15}	1	1	0	U_{48}
1	0	U_8	**1**	**1**	**1**	$\boldsymbol{U_{63}}$
			1	**0**	**1**	$\boldsymbol{U_{35}}$
			1	0	0	U_{24}

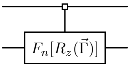
Mapping to $\mathbb{Z}$-circuit



Finding Exact Parameters (level $k = 2$)

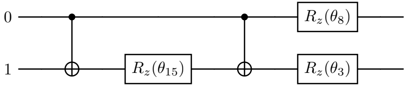
[SRBB] $\mathrm{U}^{(4)}_{\mathrm{j}^2-1} \in \mathfrak{su}(4)$ diagonal basis

$$Z(\Theta_Z) = \begin{pmatrix} e^{i(\theta_3+\theta_8+\theta_{15})} & 0 & 0 & 0 \\ 0 & e^{i(-\theta_3+\theta_8-\theta_{15})} & 0 & 0 \\ 0 & 0 & e^{i(\theta_3-\theta_8-\theta_{15})} & 0 \\ 0 & 0 & 0 & e^{i(-\theta_3-\theta_8+\theta_{15})} \end{pmatrix} \equiv \begin{pmatrix} R_z(2\gamma_1) & 0 \\ 0 & R_z(2\gamma_2) \end{pmatrix} =$$



Decimal	Binary	String	Matrix	Element
1	01	$I_2 \otimes \sigma_3$	diag(1,-1,1,-1)	3
2	10	$\sigma_3 \otimes I_2$	diag(1,1,-1,-1)	8
3	11	$\sigma_3 \otimes \sigma_3$	diag(1,-1,-1,1)	15

(a) Pauli string representation for $\mathcal{J}_Z$ elements.



(b) Z-factor circuit for $n = 2$.

Linear systems for each level k

$$\boxed{M_k \vec{x}^{(k)}_{mod} = \vec{b}^{(k)}_{mod}}$$

$$\boxed{M_k = \begin{pmatrix} M_{k-1} & +1 & M_{k-1} \\ M_{k-1} & -1 & -M_{k-1} \end{pmatrix}}$$

Solution: only M_{k-1} is necessary

θ_3	θ_8	θ_{15}
+	+	+
−	+	−
+	−	−
−	−	+

(a) M_2.

θ_3	θ_8	θ_{15}	θ_{24}	θ_{35}	θ_{48}	θ_{63}
+	+	+	+	+	+	+
−	+	−	+	−	+	−
+	−	−	+	+	−	−
−	−	+	+	−	−	+
+	+	+	−	−	−	−
−	+	−	−	+	−	+
+	−	−	−	−	+	+
−	−	+	−	+	+	−

(b) M_3.

Testing E-SRBB

GitHub: qslab-unipr/eqsp-srbb

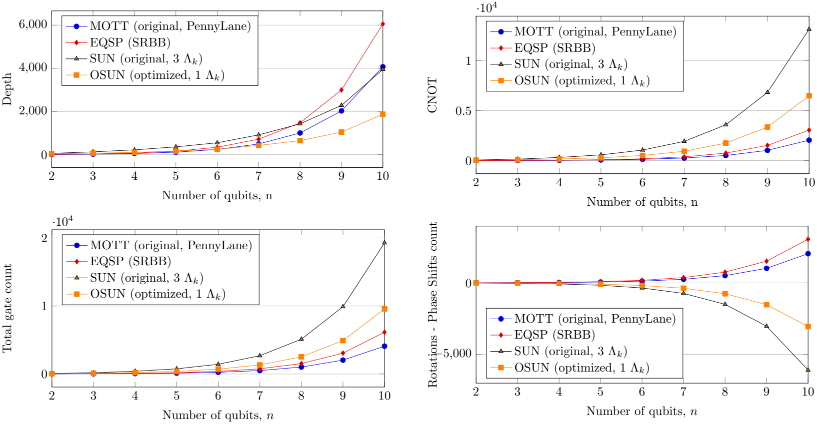
Simulation results for specific quantum states

n	Target State	Fidelity	Depth	CNOT	R_y, R_z	Time Classical (s)
2	Bell Φ_+	$\mathbf{1-3\times10^{-16}}$	**8**	2	4	6.33×10^{-4}
	Bell Φ_-	$1.00+\mathcal{O}(10^{-16})$	**12**	4	7	7.02×10^{-4}
	Bell Ψ_+	$1-3\times10^{-16}$	8	2	4	6.88×10^{-4}
	Bell Ψ_-	$1.00+\mathcal{O}(10^{-16})$	12	4	7	7.29×10^{-4}
3	GHZ	$1-9\times10^{-16}$	19	8	11	1.08×10^{-3}
	W	$\mathbf{1-2\times10^{-15}}$				1.10×10^{-3}
	Dicke	$1-2\times10^{-15}$				1.18×10^{-3}
4	GHZ	$1-2\times10^{-15}$	45	22	26	2.19×10^{-3}
	W	$1-2\times10^{-15}$				2.14×10^{-3}
	Dicke	$1-2\times10^{-15}$				2.12×10^{-3}

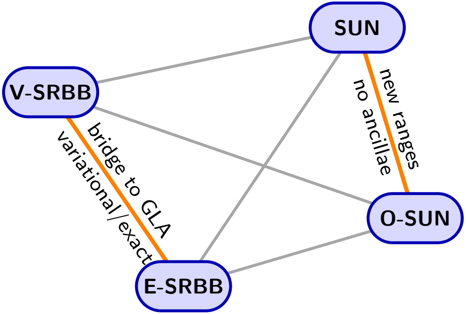
Simulation results for a sample of 20 random states with complex amplitudes

n	Type of State	Avg Fidelity $\sigma^2 \in [10^{-31}, 10^{-29}]$	Depth	CNOT	R_y, R_z	Avg Time Classical (s)
7	dense	$1-2\times10^{-15}$	716	366	374	1.16×10^{-2}
8	dense	$1-3\times10^{-15}$	1473	748	757	2.50×10^{-2}
9	dense	$1-3\times10^{-15}$	2997	1514	1524	4.79×10^{-2}
10	dense	$1-3\times10^{-15}$	6056	3048	3059	9.28×10^{-2}
11	dense	$1-6\times10^{-15}$	12186	6118	6130	1.64×10^{-1}
12	dense	$\mathbf{1-3\times10^{-15}}$	24459	12260	**12273**	$\mathbf{2.75\times10^{-1}}$

Overall Evaluation



Complete Map and Future Directions



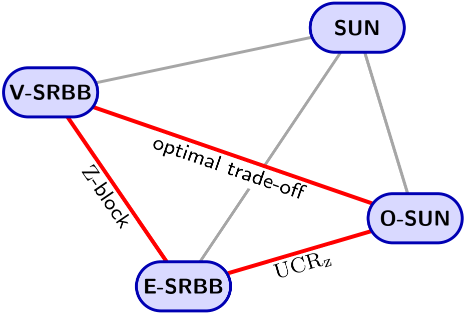
Edge SRBB

- QNN parallelization
- Sparse states reduction
- New variational metrics
- Graphical Linear Algebra

Edge SUN

- Extended ancillary range
- No-ancillae version

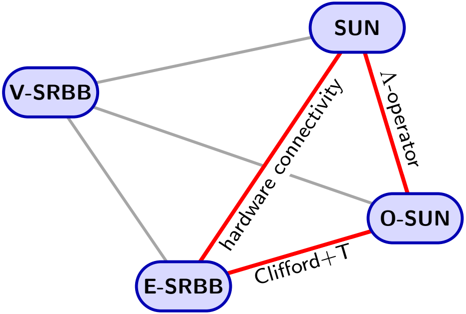
Complete Map and Future Directions



Link SRBB←SUN

- SRBB with ancillae
- UCG post Gray-Code

Complete Map and Future Directions



Link SRBB↔SUN

- Fault tolerant gate set
- Connectivity constraints

Thank You!

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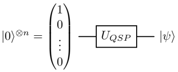
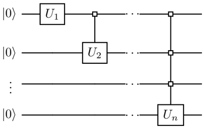
Parameter computation - Exact baseline (SUN)



$\Rightarrow$ only the first column is relevant

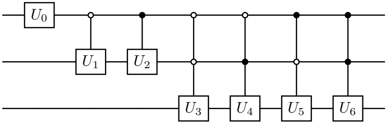


$\Rightarrow$ recursive pattern on the first column

Example for $n = 3$

$$\begin{pmatrix}
a_3 \cdot a_1 \cdot a_0 & \dots & \dots \\
b_3 \cdot a_1 \cdot a_0 & \dots & \dots \\
a_4 \cdot b_1 \cdot a_0 & \dots & \dots \\
b_4 \cdot b_1 \cdot a_0 & \dots & \dots \\
a_5 \cdot a_2 \cdot b_0 & \dots & \dots \\
b_5 \cdot a_2 \cdot b_0 & \dots & \dots \\
a_6 \cdot b_2 \cdot b_0 & \dots & \dots \\
b_6 \cdot b_2 \cdot b_0 & \dots & \dots
\end{pmatrix}$$

$$\begin{cases} a_j = \cos\gamma_j \cdot e^{i(\alpha_j - \beta_j - \delta_j)} \\ b_j = \sin\gamma_j \cdot e^{i(\alpha_j + \beta_j - \delta_j)} \end{cases}$$

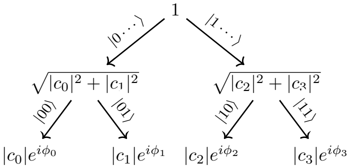


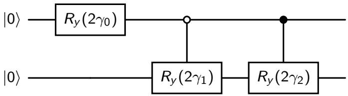
$$\begin{cases} (\vec{\alpha}, \vec{\beta}, \vec{\delta}) \leftrightarrow \Lambda_n^{(1)}, \Lambda_n^{(3)} \Rightarrow \text{linear system} \\ \vec{\gamma} \leftrightarrow \Lambda_n^{(2)} \Rightarrow \text{BST diagram} \end{cases}$$

Parameter computation - Part 2

BST diagram for **magnitudes**

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$





$$\gamma_0 = \arccos\sqrt{\frac{|c_0|^2+|c_1|^2}{1}}$$

$$\gamma_1 = \arccos\sqrt{\frac{|c_0|^2}{|c_0|^2+|c_1|^2}} \quad \dots$$

Linear system for **complex phases**

First column: phase parameters

$\downarrow$

$$\begin{cases} \alpha_j \leftrightarrow x_j \\ \beta_j \leftrightarrow y_j \\ \delta_j \leftrightarrow z_j \end{cases}$$

$\downarrow$

$2^n \times n$ real matrix

$\downarrow$

$3 \cdot (2^n - 1) - 2^n$ free parameters

CNOT-optimized Z-factor ($n \geqslant 3$)

Diagonal elements: $\zeta_n = \{j^2 - 1,\ 2 \leqslant j \leqslant 2^n\}$. Example for $n = 3$:

Position	1° q	2° q	3° q	Element	m	m'
1	0	0	1	U_3	3	
2	0	1	0	U_8	2	
3	0	1	1	U_{15}	3	2
4	1	0	0	U_{24}	1	
5	1	0	1	U_{35}	3	1
6	1	1	0	U_{48}	2	1
7	1	1	1	U_{63}	3	1,2

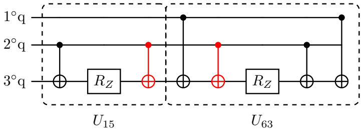


Figure: CNOT-simplification (red) for the sub-sequence $U_{15} - U_{63}$.

$n=1$	$n=2$		$n=3$			Element
1° q	1° q	2° q	1° q	2° q	3° q	
			0	0	0	-
			0	0	1	U_3
	0	0	0	1	1	U_{15}
0	0	1	0	1	0	-
1	1	1	1	1	0	-
	1	0	1	1	1	U_{63}
			1	0	1	U_{35}
			1	0	0	-

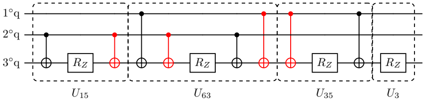
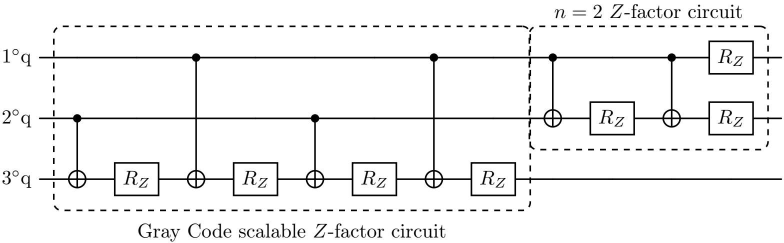


Figure: Simplifications (red) coming from new contributions only.

[G. Belli et al., IEEE QCE24, Vol. 02. 49–54, 2024]

CNOT-optimized Z-factor ($n = 3$)



- The general case n is composed by two blocks
- First block (**scalable**): Gray Code pattern encoding the optimal ordering
- Second block (**recursive**): Z-factor circuit for the case $n - 1$

[G. Belli et al., IEEE QCE24, Vol. 02. 49–54, 2024]

Testing V-SRBB

Paper: arXiv:2503.13647
GitHub: qslab-unipr/aquaman-sp

HPC Simulation for Random Quantum States

n	Time (Adam)	Error (Adam+F)	Error (Adam+T)	Time (NM)	Error (NM)
2	$\sim 9m$	$\sim 10^{-15}$	$\sim 10^{-3}$	$\sim 2s$	$\sim 10^{-15}$
3	$\sim 16m$	$\sim 10^{-13}$	$\sim 10^{-3}$	$\sim 30s$	$\sim 10^{-15}$
4	$\sim 33m$	$\sim 10^{-13}$	$\sim 10^{-3}$	$\sim 9m$	$\sim 10^{-10}$
5	**∼ 1h**	$\boldsymbol{\sim 10^{-7}}$	$\boldsymbol{\sim 10^{-3}}$	–	–
6	$\sim 2h$	$\sim 10^{-3}$	$\sim 10^{-3}$	–	–
7	$\sim 5h$	$\sim 10^{-3}$	$\sim 10^{-3}$	–	–
8	$\sim 11h$	$\sim 10^{-3}$	$\sim 10^{-3}$	–	–

Isometry $|\psi\rangle \to |\phi\rangle$ on Real Hardware (shots)

n	Rigetti Ankaa-3 (1024)	Rigetti Ankaa-3 (4096)	Rigetti Ankaa-3 (8192)	IQM Garnet (1024)	**IonQ Forte1 (1024)**
2	∼ 0.2	∼ 0.19	∼ 0.19	∼ 0.06	∼ **0.05**
3	∼ 0.6	∼ 0.52	∼ 0.51	∼ 0.4	∼ **0.09**
4	∼ 0.6	∼ 0.55	∼ 0.6	∼ 0.44	∼ **0.19**

Isometry $|\psi\rangle \to |\phi\rangle$

n	Error (Adam+F)	Error (NM)
2	$\sim 10^{-14}$	$\sim 10^{-15}$
3	$\sim 10^{-13}$	$\sim 10^{-15}$
4	$\boldsymbol{\sim 10^{-13}}$	–
5	$\sim 10^{-4}$	–
6	$\sim 10^{-3}$	–
7	$\sim 10^{-3}$	–

Specific Quantum States on Real IQM Hardware

State $(c_0, c_1, ..., c_{2^n-1})$	n	Hellinger distance (Deneb)	**Hellinger distance (Garnet)**
Bell	2	∼ 0.13	∼ **0.12**
$(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$	2	∼ 0.04	∼ **0.09**
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	2	∼ 0.01	∼ **0.01**
GHZ	3	∼ 0.38	∼ **0.31**
dense, uniform $c_i = \frac{1}{\sqrt{8}}$	3	∼ 0.1	∼ **0.06**
dense, uniform $c_i = \frac{1}{\sqrt{16}}$	4	∼ 0.27	∼ **0.14**
$(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0)$	4	∼ 0.33	∼ **0.32**