A Reversible Crumbling Abstract Machine for Plotkin's Call-By-Value

Nicolò Pizzo, Claudio Sacerdoti Coen
University of Bologna
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Introduction

λ-terms

We can see λ-terms as:

	Programs	**Proofs**
β-step		
β^{-1}-step		

The computational operation on λ-terms is **β-reduction**

λ-terms

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What does it mean to revert a β-step?

λ-terms

We can see λ-terms as:

	Programs	Proofs
β-step	Computing function application	Simplifying a term of the proof
β^{-1}-step	Restoring the previous state	Backtracking simplification

The computational operation on λ-terms is **β-reduction**

What does it mean to revert a β-step?

The goal

We want to equip Plotkin's **weak**, **closed** CbV calculus with **reversibility**

Plotkin's CbV calculus (λ_{Plot})

Terms	$u, t ::= v \mid ut$
Values	$v ::= \lambda x.t \mid x$
Right v-context	$R ::= \langle\cdot\rangle \mid tR \mid Rv$
Reduction Rule	$R\langle(\lambda x.t)v\rangle \to_\beta R\langle t\{x \leftarrow v\}\rangle$

We fix the right-to-left evaluation order

Landauer's embedding

We introduce some auxiliary structure, storing the necessary information to revert each step of the computation.

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Example

$(x, y) \mapsto x + y$

Landauer's embedding

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Example

$(x, y) \mapsto x + y$

$(x, y) \mapsto (x + y, x)$

Landauer's embedding for λ_{Plot}

λ_{Plot} Reduction rule

$$R\langle(\lambda x.t)v\rangle \to_\beta R\langle t\{x \leftarrow v\}\rangle$$

Examples

$t = (\lambda x.xx)(\lambda y.y)$	$\to_\beta (\lambda y.y)(\lambda y.y)$	Remember $\lambda x.xx$
$t = (\lambda x.y)M$	$\to_\beta y$	Remember M

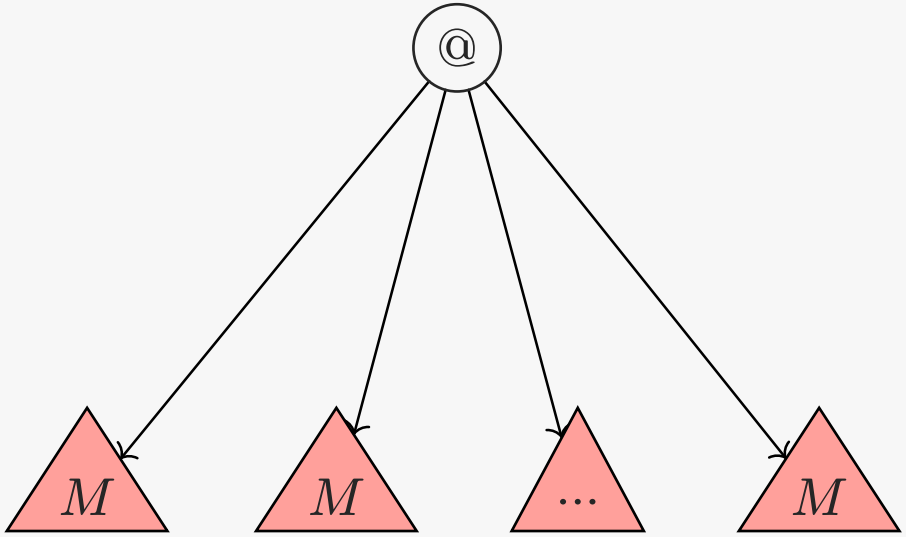
The cost is $O(|t|)$ **for each step**

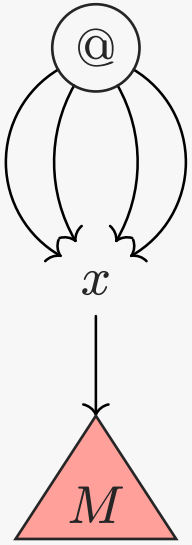
The size-explosion problem

$$(\lambda x.xx \cdots x)M \quad \rightarrow_\beta \quad MM \cdots M$$

If M is similar in size to $(\lambda x.xx \cdots x)M$, the term size explodes!

Sharing





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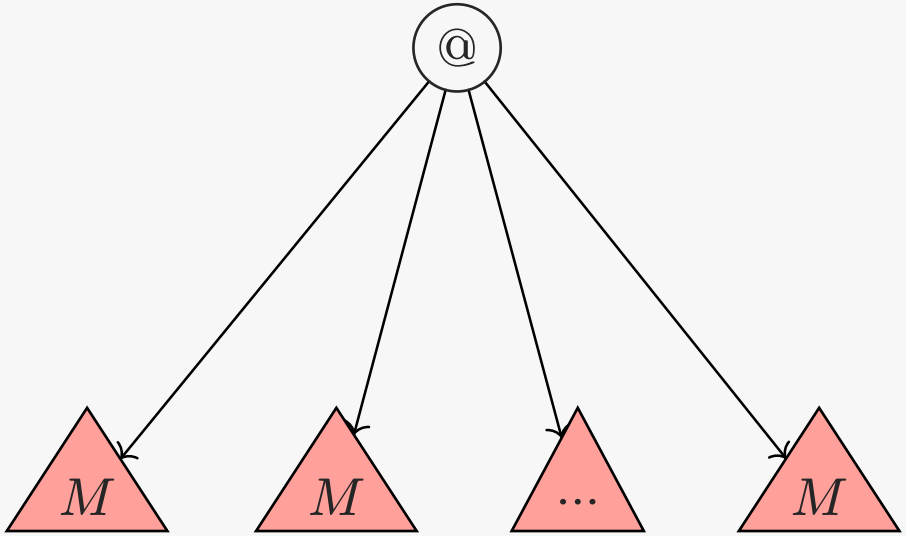

Let-in

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let x = M in x x ... x
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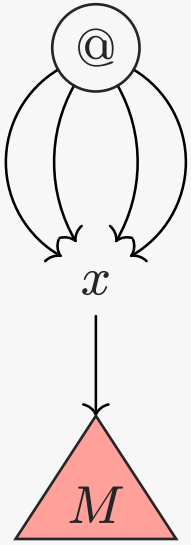

Explicit Substitutions

$(xx \cdots x)[x \leftarrow M]$

Implementing λ-calculi

Abstract machines allow for efficent implementations of λ-calculi

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Implementing λ-calculi

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- SECD Machine
  - “A Reversible SE(M)CD Machine”, W. Kluge 1999
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Implementing λ-calculi

Abstract machines allow for efficent implementations of λ-calculi

- SECD Machine
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- **Crumbling Machines**

Crumbling

With **crumbling** we avoid nested applications

Example

$$(\lambda x.x(xx))(\lambda y.y)$$

Sharing: $x(xx)[x \leftarrow \lambda y.y]$

Crumbling: $(z_1 z_2)[z_1 \leftarrow \lambda x.(xz_3)[z_3 \leftarrow xx]][z_2 \leftarrow \lambda y.y]$

Crumbling Abstract Machines

Crumbling Abstract Machines encode both the *evaluation order* and the *evaluation context*!

Example

λ-term	$II \quad (v_1 v_2) \quad v_3$
Right-to-left	$(z_1 z_2)[z_1 \leftarrow \underline{I}][z_2 \leftarrow z_3 z_4][z_3 \leftarrow \underline{I}][z_4 \leftarrow z_5 z_6] \quad [z_5 \leftarrow z_7 z_8] \quad \left[z_7 \leftarrow \underline{v_1}\right]\left[z_8 \leftarrow \underline{v_2}\right]\left[z_9 \leftarrow \underline{v_3}\right]$
Left-to-right	$(z_1 z_2)[z_2 \leftarrow z_3 z_4]\left[z_4 \leftarrow \underline{v_3}\right][z_3 \leftarrow z_5 z_6]\left[z_6 \leftarrow \underline{v_2}\right]\left[z_5 \leftarrow \underline{v_1}\right] \quad [z_1 \leftarrow z_7 z_8] \quad [z_8 \leftarrow \underline{I}][z_7 \leftarrow \underline{I}]$

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The RCAM

Crumbled calculi

- “Crumbling Abstract Machines”, *B. Accattoli et al., 2019*
- “Positive Sharing and Abstract Machines”, *B. Accattoli et al., 2025*
- Finest Crumbling,

Crumbled calculi

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- Finest Crumbling, **N. Pizzo and C. Sacerdoti Coen, 2026**

Finest Crumbling and RCAM

Plotkin Terms

Terms	Values
$u, t ::= v \mid ut$	$v ::= \lambda x.t \mid x$

Finest Crumbling

Bites	Crumbled Values	Environments
$b ::= xy \mid v$	$v ::= \lambda x.[* \leftarrow y] \mid \lambda x.E$	$E ::= \varepsilon \mid E[z \leftarrow b]$

Finest Crumbling and RCAM

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RCAM

State	History Stack
$S ::= (E, E_v, \mathcal{H})$	$\mathcal{H} ::= \varepsilon \mid \langle\rangle : \mathcal{H} \mid \langle x, y\rangle : \mathcal{H}$

Finest Crumbling and RCAM

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$\iota(\cdot)\downarrow$ $\uparrow_{\cdot\Downarrow}$

RCAM

State	History Stack
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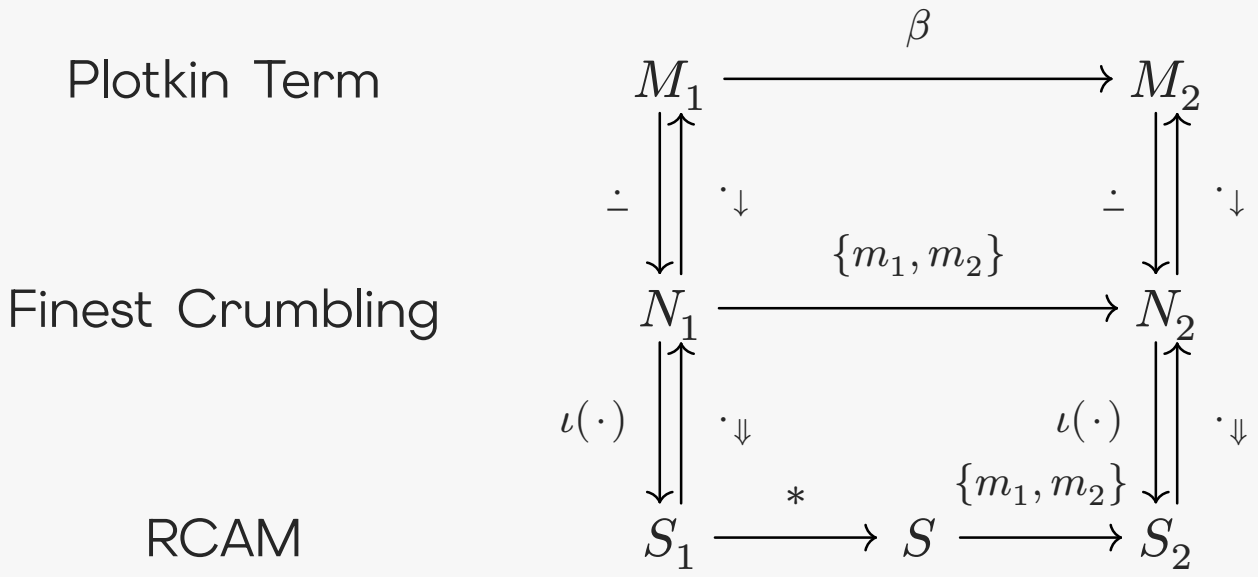
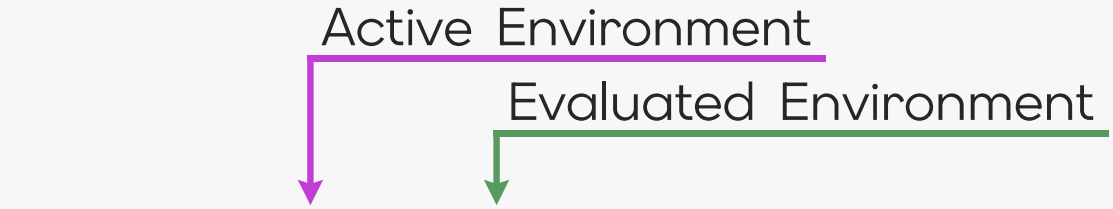
Finest Crumbling and RCAM



RCAM Transitions



$$(E_1[z \leftarrow xy], E_v, \mathcal{H}) \;{}_{m_1^b}\!\!\leftrightsquigarrow_{m_1}\; (E_1[z \leftarrow b']E', E_v, \langle x, y\rangle : \mathcal{H})$$

$$\text{Where } E_v(x) = \lambda x.[* \leftarrow b]E, \quad [* \leftarrow b]E\{x \leftarrow y\} = [* \leftarrow b']E'$$

$$(E_1[z \leftarrow xy], E_v, \mathcal{H}) \;{}_{m_2^b}\!\!\leftrightsquigarrow_{m_2}\; (E_1, [z \leftarrow E_v(y')]\; E_v, \langle x, y\rangle : \mathcal{H})$$

$$\text{Where } E_v(x) = \lambda x.[* \leftarrow x'], \quad [* \leftarrow x']\{x' \leftarrow y\} = [* \leftarrow y']$$

$$(E_1[z \leftarrow v], E_v, \mathcal{H}) \;{}_{\text{sea}^b}\!\!\leftrightsquigarrow_{\text{sea}}\; (E_1, [z \leftarrow v]E_v, \langle\rangle : \mathcal{H})$$

Reverting a step: the idea

$$(E_1[z \leftarrow xy], E_v, \mathcal{H}) \rightsquigarrow_{m_1} (E_1[z \leftarrow b']E', E_v, \langle x, y\rangle : \mathcal{H})$$

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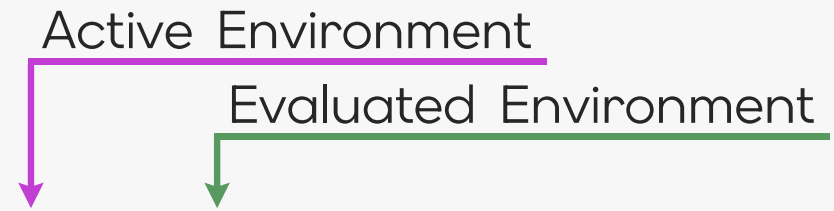
Reverting a step: the idea



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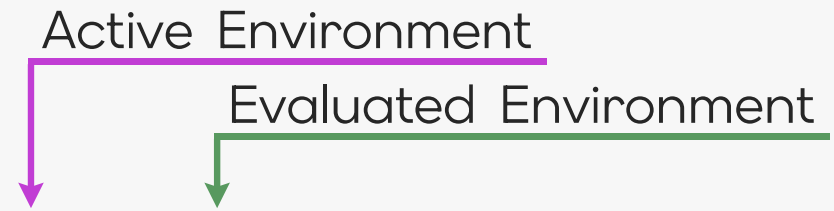

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$$(E_1[z \leftarrow \not{b}']E', E_v, \langle x, y\rangle : \mathcal{H}) \rightsquigarrow_{m_1^b} (E_1[z \leftarrow xy], E_v, \mathcal{H})$$

xy

Reverting a step: the idea

Active Environment

Evaluated Environment

$$(E_1[z \leftarrow xy], E_v, \mathcal{H}) \rightsquigarrow_{m_1} (E_1[z \leftarrow b']E', E_v, \langle x, y\rangle : \mathcal{H})$$

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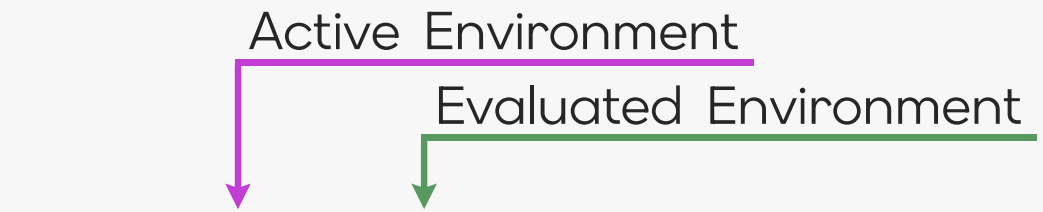
xy

$E_v(x)$ already holds this information

Reverting a step: the idea



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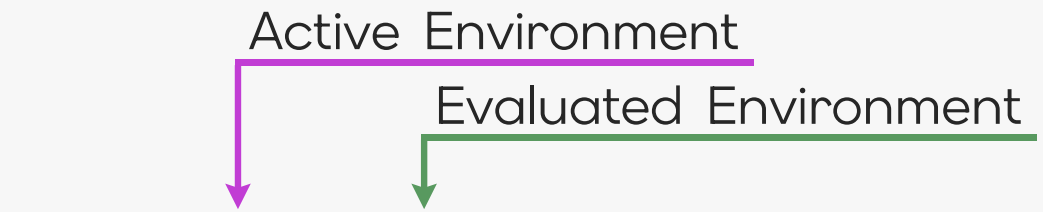

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xy

$E_v(x)$ already holds this information

We just need to save the pair x, y with cost $O(1)$

Main Results

Theorem (Implementation)

The RCAM implements $\rightarrow_\beta$-evaluation.

Theorem (The RCAM is bi-linear)

For any closed term t, and any RCAM execution $\rho : \iota(\underline{t}) \rightsquigarrow_f^* S$, the cost of implementing ρ on a RAM is $O((|\rho|_p + 1) \cdot |t|)$.

Theorem (Space overhead)

For any closed term t, and any RCAM execution $\rho : \iota(\underline{t}) \rightsquigarrow_f^* S$, the maximum size of the history during the execution ρ on a RAM is $O((|\rho|_p + 1) \cdot |t|)$

Main Results

Theorem (Backward Determinism)

The transitions $\rightsquigarrow_b$ are deterministic.

Theorem (Loop)

Let S' b any state of the RCAM, such that $S \rightsquigarrow_f S'$. Then the evaluation step can always be reverted, i.e. $S' \rightsquigarrow_b S$ and vice-versa.

Theorem (Well-foundedness)

There is no infinite reverse computation, i.e. we do not have sequences $(S_i)_{i \in \mathbb{N}}$ such that $S_{i+1} \rightsquigarrow_f S_i$ for all $i \in N$.

Conclusions

Summary

- Explored Finest Crumbling
- Efficient Reversible Crumbling Abstract Machine
- Compact Landauer's embedding

Future Works

- Open/Strong Call-by-Value
  - Interesting applications to ITPs
- Scalability to Call-by-Name and Call-by-Need

Thank you!