

Quantum alternation

Prakash Panangaden¹

¹School of Computer Science
McGill University and
School of Informatics
University of Edinburgh

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Outline

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- 2 Basic background

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- Measurement calculus: low-level, close to implementation.
- Selinger's Quantum Programming Language: Quantum data and classical control.
- There are many more: Quipper, Qwire, Q#,....

Example

Simple program

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input  $b$ :bit;  
input  $p, q$ :qbit;  
 $b := \text{measure } p$ ;  
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- The conditional is based on a classical boolean.

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- Quantum alternation is problematic in general.

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- Any time we say “operator” we mean “bounded linear (= continuous) operator.”

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- A **positive map** is a map from $\mathcal{B}(\mathcal{H})$ to itself such that it takes positive operators to positive operators.

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- A **superoperator** is a cp map that is also trace non-increasing.

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- Choi: The action of $\phi \in CP(M_n, M_k)$ can be given explicitly as a matrix in M_{nk} depending on the particular Kraus decomposition.
- Stinespring: For any completely positive map $\theta : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ there is a triple $(\pi, V, \mathcal{K})$ where $\mathcal{K}$ is a Hilbert space, $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$ is a representation and $V : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$\theta(a) = V^\dagger \pi(a) V.$$

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- If $\theta \in CP(M_n, M_k)$ then the minimal Stinespring representation is in M_m where $m \leq n^2 k$.

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- By Stinespring, there exists an ancilla $\mathcal{A}$ and an operator $V : \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{A}$ such that

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- Easy to check $\mathcal{E}(\rho) = \sum_{i=1}^k V_i^* \rho V_i$.
- The V_i give a Kraus representation for $\mathcal{E}$.

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- A function from a dcpo to another dcpo is called **Scott continuous** if it preserves lubs of increasing sequences.

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- 4 measure q [measure and branch on the result]
- 5 merge [combine two possible paths]

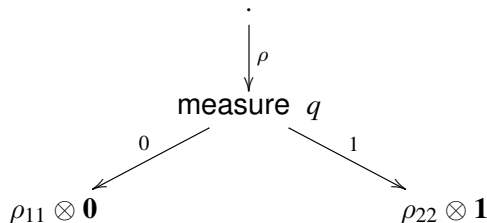
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- discard q ; $\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \mapsto [\rho_{11} + \rho_{22}]$

Semantics of measurement



Here $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

$$\rho_{11} \otimes \mathbf{0} = \begin{pmatrix} \rho_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\rho_{22} \otimes \mathbf{1} = \begin{pmatrix} 0 & 0 \\ 0 & \rho_{22} \end{pmatrix}$$

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- This sum can be proven to converge yielding a density matrix with trace ≤ 1 .

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- Because the density matrices form a dcpo we are sure that the lubs exist.
- Recursion can implement iteration but not the other way around.

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- Suppose we have a qubit q and two superoperators $S, T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ then the quantum alternation $(qAlt)(q; S, T)$ should be a superoperator from $\mathcal{B}(\mathcal{Q} \otimes \mathcal{H}) \rightarrow \mathcal{B}(\mathcal{Q} \otimes \mathcal{K})$.

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- Toffoli gate uses nested if:
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- Very useful for describing algorithms especially if there are only unitaries.

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- can we extend it to quantum operations that are not unitaries?

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- A basis corresponds to a particular choice of measurement. Thus the particular Kraus representation is dictated by how the experimenter chooses to describe the environment.

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- We give compositional semantics but in terms of specific choices of Kraus operators, we do not try to give compositional superoperator semantics.

Quantum alternation of unitaries

Given unitary operators U, V on $\mathcal{H}$ and a qubit q (space $\mathcal{Q}$) we define

$$|0\rangle\langle 0| \otimes U + |1\rangle\langle 1| \otimes V = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

as the quantum alternation of U and V .

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- This defines a superoperator

$$\mathcal{S}(\rho) = \sum_{i,j} K_{i,j}^* \rho K_{i,j}.$$

What Stinespring says

If one looks at the Stinespring dilation corresponding to the above construction we see that the ancilla spaces (environments) of the two Kraus forms are tensored together.

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- Applying a unitary

$$\llbracket q^* = U \rrbracket = \{U\}.$$

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- We do not give semantics for loops and conditionals.

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- This example arose from discussions with Mingsheng Ying and Yuan Feng at UTS Sydney based on an example due to Nengkun Yu.
- One can think of quantum alternation as an algorithmic notation, it is not clear what it means *physically*.

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$$W = \begin{pmatrix} \lambda U & 0 \\ 0 & \mu V \end{pmatrix}$$

- By explicit calculation we can show that $R' \not\leq R$.

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- No discussion of what quantum alternation means physically or mathematically.

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Summary

- Quantum alternation is troublesome: non-compositional and non-monotone.
- Is it a sensible thing to even consider? It came from programming languages without thinking about physics.
- One should look at real physical situations, e.g. Mach-Zehnder interferometers, quantum random walks or adiabatic quantum computation and extract a notion of quantum alternation.
- Perhaps quantum alternation and unrestricted recursion is not allowed in nature!

Thank you!